

Practical Guide to the Symbolic Computation of Symmetries of Differential Equations ¹

Stanly Steinberg, Professor
Department of Mathematics and Statistics
University of New Mexico
Albuquerque, New Mexico, 87131, USA
E-mail: stanly@math.unm.edu

Rubens de Melo Marinho Junior, Professor
Physics Department
Instituto Tecnológico de Aeronáutica
Praça Mal. Eduardo Gomes
50 - Vila das Acácias
São José dos Campos - SP, 12228-900, Brazil
E-mail: marinho@ita.br or marinho.rubens@gmail.com

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Chapter 1

TRANSFORMATION GROUPS

1.1 Introduction

The concept of symmetry is basic to mathematics and its applications and has many interpretations for many different objects. In this monograph, the objects under discussion will be systems of linear and nonlinear ordinary or partial differential equations. A symmetry will be a mapping of the solution space of the system into the solution space of the system, that is, a mapping that leaves the solution space invariant. In the first few chapters the mapping will be restricted to transformations that involve both the independent and dependent variables used in the system of differential equations. This idea of a symmetry includes, as a special case, a change of variables that maps a differential equation into itself, that is, leaves the differential equation invariant. The observation that rotations leave Laplace's equation invariant then provides an example of the type of symmetries that will interest us. Later, the notion of a symmetry will be generalized to mappings that depend on the derivatives of the solutions of the system of differential equations.

Many systems of differential equations have obvious symmetries that involve translations, scalings of the variables, rotations and other geometric transformation. Most such symmetries can be found by inspection or by doing some elementary calculation. We will be interested in problems that have *hidden symmetries*, that is, symmetries that cannot be found using elementary techniques. Of course, our method will produce the elementary as

well as the hidden symmetries.

The purpose of this paper is the description of a set of computer programs written in the MACSYMA/VAXIMA symbol manipulation language. For elementary examples, one of these programs uses, as input, a description of the differential equation and produces, as output, the infinitesimal symmetries. For more complicated examples, the programs need some human help and thus the programs are designed to make such human intervention easy. Because of the precision demanded by the computer coding, we have devoted the first chapter of this monograph to a careful description of the methods that we use to compute the symmetries. This discussion was based on a minimum of mathematical concepts in an attempt to make this material accessible to the widest possible audience.

Perhaps one of the greatest impediments to the use of symmetries is the volume of trivial algebra required to investigate even a modest problem. We believe we have provided software that reduces this algebra to an absolute minimum! Once the symmetries have been computed, then it is important to use the symmetries to obtain some useful information about differential equations. No applications are included in this monograph. However, a perusal of our references should convince the reader that there are extensive applications of these ideas. We have included some programs to help with applications. Included are programs to transform differential equations to new coordinate frames and programs to help calculate similarity solutions of differential equations.

In principle, the symmetries of a system can be calculated by assuming a general form for the symmetry and then solving the invariance condition for the symmetry. In practice this problem is intractable. It has been discovered that these computational difficulties can be overcome by restricting the notion of symmetry still further. The crucial concept is that of a one parameter group and its infinitesimal. Recall that two transformations can be applied sequentially to form a new transformation which is called the composition of the two transformations. Because only invertible transformations are considered, the composition of transformations forms an algebraic group. This explains one word in the terminology. A one parameter group of transformations is a set of transformations that depend on one real parameter and, moreover, addition in the parameter is equivalent to composition of the transformations. The problem of directly computing the one parameter groups is as intractable as computing a single transformation.

The idea that is responsible for the power of the group method is to dif-

ferentiate with respects to the parameter. The derivative, at the origin, with respects to the parameter produces the infinitesimal group. The derivative, at the origin, of the invariance condition with respects to the parameter yields a system of *linear* partial differential equations for the infinitesimal symmetry. These equations are called the *determining* equations. If these equations can be solved for the infinitesimal symmetries, then the one parameter groups can be found from the infinitesimal groups by solving a system of ordinary differential equations. The process of deriving the one parameter group from the infinitesimal group is called *exponentiation*. The reasons for this terminology will become clear later. The process of finding the infinitesimal symmetries is called a mess.

Before the difficulties of finding the infinitesimal symmetries are described, we note that for many problems that are intuitively symmetric, the infinitesimal and one parameter groups of symmetries can, in fact, be computed. On the other hand, it is worth noting that the, discrete symmetries such as the transformations that are reflections in one axis are *not* computable by these methods. Even though not all important symmetries are computable by these methods, the symmetries that can be computed have many important applications.

As was noted above, the problem of finding the symmetries of a system of *nonlinear* (or linear) ordinary or partial differential equations becomes a problem of solving a system of *linear* partial differential equations. The system of partial differential equations that determine the infinitesimal symmetries tends to have the following properties. The system of equations is large and overdetermined, that is, there are more equations than unknowns. Because the equations are linear their solutions form a linear space. Thus the infinitesimal symmetries form a linear space. Usually these linear spaces are essentially finite dimensional. There are theorems to this effect for some problems where the original equation is linear, see Chandler [56] and Ovsiannikov [18].

We believe that one reason that these methods have not seen more applications is that the problems of deriving the determining equations and then solving equations for the infinitesimal symmetries is lengthy and tedious. Other authors have provided programs to compute the determining equations [27, 28, 30, 31, 32, 33], however, we believe that we were the first to add programs for solving the determining equations [33]. Because of the use of computers it is essential that all definitions be clear and that all computational procedures be described precisely. On the other hand, questions

of differentiability of functions, convergence of series, global definition of objects and many other favorite questions of analysts play no significant role in the problems of computing symmetries.

If the reader is interested in some of the technical assumptions, then it is worth noting that the material that is discussed here is local, that is, all computations are carried out in some sufficiently small neighborhood of a given point. The assumption that all functions are analytic in a neighborhood of the given point will cover most of the applications that are of interest. If we are considering only neighborhoods of points, then the precise definitions of the transformation group is quite technical because the transformation group may move the given point or may move the neighborhood that is being considered. On the other hand, it is not possible to assume that the transformation groups are defined on all of Euclidean space because some of the more interesting groups have singularities.

These difficulties are not of great concern for us because we are interested in computing the infinitesimal symmetries and the infinitesimal symmetries are well defined in the neighborhood of some given point. Thus we may take the point of view that the one parameter groups are used to *motivate* a careful definition of the infinitesimal groups and thus a technically precise definition of one parameter group is not needed. When we need to compute a one parameter group from an infinitesimal group we will do this by solving a system of ordinary differential equations and consequently the theory of ordinary differential equations provides a firm foundation for these calculations.

One place where the standard mathematical terminology is not sufficiently precise is in ordinary differential equations. Thus, in the simplest situation of one first order ordinary differential equation, the equation is frequently written

$$\frac{dy}{dx} = a(x, y) \tag{1.1.1}$$

and the solution is written

$$y = y(x). \tag{1.1.2}$$

Here the letter y stands both for a real variable and the function that is to be the solution of the differential equation. Such minor abuses of notation will blow away our computer programs. Consequently we will change the notation as follows. Let $a(x, y)$ be a function, the variables (x, y) and $y = f(x)$ be a

function of x . The differential equation is then written

$$\frac{df(x)}{dx} = a(x, f(x)). \quad (1.1.3)$$

We hope the reader will bear with such fine tuning of the notation.

This monograph assumes that the reader is familiar with multivariate calculus, linear algebra, ordinary differential equations and partial differential equations. Some of the material needed from these subjects are reviewed in the text or in the appendices. It is not assumed that the reader has a background in rigorous analysis or Lie group theory. Not requiring Lie group theory as background distinguishes this development from many other developments. The use of the computer codes requires some familiarity with the MACSYMA/VAXIMA symbol manipulation programs.

The general organization of this monograph is as follows. The remainder of Chapter 1 is devoted to introducing notation, developing background material and setting up the procedure for calculating symmetries. Reviews of some well known material that is particularly important for our discussion is presented in the appendices. Chapter 2 covers the calculation of symmetries of ordinary differential equations. Chapter 3 is independent of Chapter 2 and describes the calculation of the symmetries of partial differential equations. In Chapter 4 the symmetries are allowed to depend on the derivatives of the of the solutions of the differential equations being studied. We call such transformations *jet* transformations although they are frequently called Lie-Backlund transformations. These transformations include the classical contact transformations.

The remainder of Chapter 1 is organized as follows. Section 2 sets up the notation and introduces one parameter groups of point transformations and their infinitesimals. At this point there is no need to distinguish between dependent and independent variables so they are thought of as a single set of variables. Point transformations induce a natural action on functions of the variables. In Section 3 *Lie series* are used to describe this action. The solutions of systems of differential equations are curves, surfaces or hyper surfaces in higher dimensional spaces. Section 4 describes the action of one parameter groups and their infinitesimal on curves and surfaces. Once the infinitesimal symmetries for a system of differential equations are found then many applications hinge on finding invariant functions and canonical coordinates for the group so this is discussed in Section 5. Finally, in Section 6, we describe the method for calculating the infinitesimal symmetries of a system

of differential equations.

1.2 One Parameter Groups and Infinitesimal Groups

In this section a one parameter group of transformations and the infinitesimal group of a one parameter group of transformations are defined. Also, it is shown that the infinitesimal group determines the one parameter group. The transformations will map points in n dimensional Euclidean space R^n into itself. In R^n we will denote points by

$$\mathbf{v} = (x_1, x_2, \dots, x_n), \boldsymbol{\nu} = (\xi_1, \xi_2, \dots, \xi_n). \quad (1.2.4)$$

Of course, x_i and ξ_i are real variables. A transformation will be denoted by a capital letter, say \mathbf{G} , and then we can write

$$\boldsymbol{\nu} = \mathbf{G}(\mathbf{v}), \nu_i = G_i(x_1, \dots, x_n). \quad (1.2.5)$$

Many of our examples will be given in low dimensional spaces so we introduce some special notation. If $n = 1$, then we will frequently use the notation

$$\xi = G(x) \quad (1.2.6)$$

while if $n = 2$, we will frequently use

$$\begin{aligned} \mathbf{v} &= (x, y), \boldsymbol{\nu} = (\xi, \eta) \\ \boldsymbol{\nu} &= \mathbf{G}(\mathbf{v}), \\ \xi &= G_1(x, y), \eta = G_2(x, y) \end{aligned} \quad (1.2.7)$$

and if $n = 3$, we will frequently use

$$\begin{aligned} \mathbf{v} &= (x, y, z), \boldsymbol{\nu} = (\xi, \eta, \zeta), \\ \boldsymbol{\nu} &= \mathbf{G}(\mathbf{v}), \\ \xi &= G_1(x, y, z), \eta = G_2(x, y, z), \zeta = G_3(x, y, z). \end{aligned} \quad (1.2.8)$$

We do not always use these letters for variables, but we always use the same style for labeling points.

A one parameter group of transformation is a set of transformations that depends on one real parameter and the dependence on this parameter has certain *exponential* properties. We usually denote the parameter by ϵ , the group of transformations by $\mathbf{G}(\epsilon)$ and the action of the group on points by

$$\nu(\epsilon) = \mathbf{G}(\epsilon)(\mathbf{x}) = \mathbf{G}(\epsilon, \mathbf{x}). \quad (1.2.9)$$

The intuitive idea behind the group properties is given by thinking of $\mathbf{G}(\epsilon)$ as being the exponential of “something”,

$$\mathbf{G}(\epsilon) = e^{\epsilon L}. \quad (1.2.10)$$

Then

$$\begin{aligned} \mathbf{G}(0) &= e^{0L} = 1, \\ \mathbf{G}(\epsilon_1)\mathbf{G}(\epsilon_2) &= e^{\epsilon_1 L}e^{\epsilon_2 L} = e^{(\epsilon_1 + \epsilon_2)L} = \mathbf{G}(\epsilon_1 + \epsilon_2), \\ \mathbf{G}(\epsilon)\mathbf{G}(-\epsilon) &= e^{\epsilon L}e^{-\epsilon L} = e^0 = 1. \end{aligned} \quad (1.2.11)$$

These ideas will be made rigorous in the section on Lie series. For now, we simply state the group properties:

Properties.

A: $\mathbf{G}(0, \mathbf{v}) = \mathbf{v}$

B: $\mathbf{G}(\epsilon_1, \mathbf{G}(\epsilon_2, \mathbf{v})) = \mathbf{G}(\epsilon_1 + \epsilon_2, \mathbf{v})$

C: $\mathbf{G}(\epsilon, \mathbf{G}(-\epsilon, \mathbf{v})) = \mathbf{G}(-\epsilon, \mathbf{G}(\epsilon, \mathbf{v})) = \mathbf{v}$

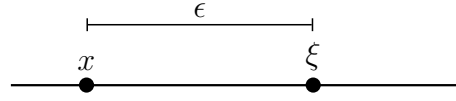
Note that property C follows from A and B by choosing $\epsilon_1 = \epsilon, \epsilon_2 = -\epsilon$ or $\epsilon_1 = -\epsilon, \epsilon_2 = \epsilon$.

Definition. A one parameter group of transformations is a set of transformations $\mathbf{G}(\epsilon, \mathbf{v})$ that depends on one real parameter and satisfies properties 1 and 2 above.

As we go along, we will illustrate each idea with two examples, translations in one variable and rotations in two variables. Later in this chapter we will discuss a rather complete set of elementary examples.

Example - Translations. Translations in one variable are given by

$$\xi = \xi(\epsilon) = G(\epsilon, x) = x + \epsilon \quad (1.2.12)$$



Translations

The group properties are satisfied:

$$G(0, x) = x$$

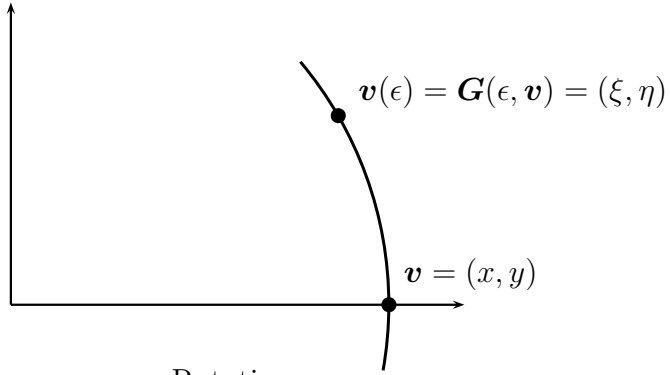
$$G(\epsilon_1, G(\epsilon_2, x)) = G(\epsilon_1, x + \epsilon_2) = x + \epsilon_2 + \epsilon_1 = G(\epsilon_1 + \epsilon_2, x)$$

Example - Rotations. Rotations in two variables are given by

$$(\xi(\epsilon), \eta(\epsilon)) = \mathbf{G}(\epsilon, x, y) = (G_1(\epsilon, x, y), G_2(\epsilon, x, y)) \quad (1.2.13)$$

where

$$\xi = \xi(\epsilon) = \cos(\epsilon)x - \sin(\epsilon)y, \eta = \eta(\epsilon) = \sin(\epsilon)x + \cos(\epsilon)y \quad (1.2.14)$$



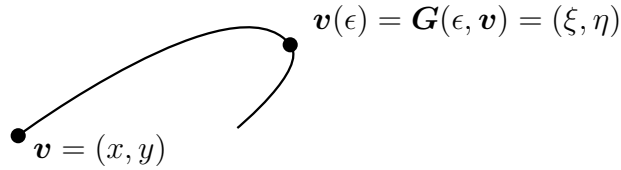
Rotations

The group properties are satisfied:

$$\begin{aligned}
\mathbf{G}(0, x, y) &= (\cos(0)x - \sin(0)y, \sin(0)x + \cos(0)y) = (x, y) . \\
\mathbf{G}(\epsilon_1, \mathbf{G}(\epsilon_2, (x, y))) &= \mathbf{G}(\epsilon_1, G_1(\epsilon_2, x, y), G_2(\epsilon_2, x, y)) \\
&= \mathbf{G}(\epsilon_1, \cos(\epsilon_2)x - \sin(\epsilon_2)y, \sin(\epsilon_2)x + \cos(\epsilon_2)y) \\
&= (\cos(\epsilon_1)[\cos(\epsilon_2)x - \sin(\epsilon_2)y] \\
&\quad - \sin(\epsilon_1)[\sin(\epsilon_2)x + \cos(\epsilon_2)y], \\
&\quad \sin(\epsilon_1)[\cos(\epsilon_2)x - \sin(\epsilon_2)y] \\
&\quad + \cos(\epsilon_1)[\sin(\epsilon_2)x + \cos(\epsilon_2)y]) \\
&= ([\cos(\epsilon_1)\cos(\epsilon_2) - \sin(\epsilon_1)\sin(\epsilon_2)]x \\
&\quad - [\cos(\epsilon_1)\sin(\epsilon_2) + \sin(\epsilon_1)\cos(\epsilon_2)]y, \\
&\quad [\sin(\epsilon_1)\cos(\epsilon_2) + \cos(\epsilon_1)\sin(\epsilon_2)]x \\
&\quad + [\cos(\epsilon_1)\cos(\epsilon_2) - \sin(\epsilon_1)\sin(\epsilon_2)]y) \\
&= (\cos(\epsilon_1 + \epsilon_2)x - \sin(\epsilon_1 + \epsilon_2)y, \\
&\quad \sin(\epsilon_1 + \epsilon_2)x + \cos(\epsilon_1 + \epsilon_2)y) \\
&= \mathbf{G}(\epsilon_1 + \epsilon_2, x, y) .
\end{aligned}$$

By the way, the group properties are obvious from geometric considerations.

We now turn to the definition of the infinitesimal of a one parameter group of transformations. It is usual to think of the infinitesimal as the transformation given by infinitely small changes in ϵ . Observe that if \mathbf{v} is fixed and ϵ is allowed to vary, then $G(\epsilon, \mathbf{v})$ is a parametric curve in R^n passing through \mathbf{v} when $\epsilon = 0$.



One parameter curve

Definition. The infinitesimal of a one parameter group is a vector field $\mathbf{T}(\mathbf{v})$ given by the tangent vector to the curve $\mathbf{G}(\epsilon, \mathbf{v})$ at $\epsilon = 0$, that is

$$\mathbf{T}(\mathbf{v}) = \frac{\partial}{\partial \epsilon} \mathbf{G}(\epsilon, \mathbf{v}) \big|_{\epsilon=0} . \quad (1.2.15)$$

Because of the group property, the tangent vector to the curve $\mathbf{G}(\epsilon, \mathbf{v})$ at any point ϵ_0 can be found in terms of the infinitesimal vector field;

$$\begin{aligned}\frac{\partial \mathbf{G}(\epsilon, \mathbf{v})}{\partial \epsilon} \big|_{\epsilon=\epsilon_0} &= \frac{\partial}{\partial \epsilon} \mathbf{G}(\epsilon + \epsilon_0, \mathbf{v}) \big|_{\epsilon=0} \\ &= \frac{\partial}{\partial \epsilon} \mathbf{G}(\epsilon, G(\epsilon_0, \mathbf{v})) \big|_{\epsilon=0} \\ &= \mathbf{T}(\mathbf{G}(\epsilon_0, \mathbf{v})).\end{aligned}\tag{1.2.16}$$

One interpretation of this last formula is that the curve $\mathbf{G}(\epsilon, \mathbf{v})$ is everywhere tangent to the direction field $\mathbf{T}(\mathbf{v})$.

Another interpretation is that the formula is, in fact, a system of ordinary differential equations for determining $\mathbf{G}(\epsilon, \mathbf{v})$ from $\mathbf{T}(\mathbf{v})$. If we think of \mathbf{v} and \mathbf{T} as given and write $d/d\epsilon$ for $\partial/\partial\epsilon$ then (1.2.16) along with the fact that $\mathbf{G}(0, \mathbf{v}) = \mathbf{v}$ gives

$$\frac{d}{d\epsilon} \mathbf{G}(\epsilon, \mathbf{v}) = \mathbf{T}(\mathbf{G}(\epsilon, \mathbf{v})), \mathbf{G}(0, \mathbf{v}) = \mathbf{v}.\tag{1.2.17}$$

This is an initial value problem that determines $\mathbf{G}(\epsilon, \mathbf{v})$ in terms $\mathbf{T}(\mathbf{v})$.

Example - Translations - Continued. The translation group in one dimension is given by

$$G(\epsilon, x) = x + \epsilon.$$

The infinitesimal group is given by

$$T(x) = \frac{d}{d\epsilon} G(\epsilon, x) \big|_{\epsilon=0} = 1.\tag{1.2.18}$$

The initial value problem for determining $G(\epsilon, x)$ in terms of $T(x)$ is

$$\frac{d}{d\epsilon} G(\epsilon, x) = 1, G(0, x) = x.\tag{1.2.19}$$

This notation for an ordinary differential is not the usual so we introduce

$$\xi = \xi(\epsilon) = G(\epsilon, x)\tag{1.2.20}$$

and then rewrite (1.2.19) as

$$\frac{d\xi}{d\epsilon} = 1, \xi(0) = x.\tag{1.2.21}$$

This can be integrated to

$$\xi = \epsilon + C \quad (1.2.22)$$

and then the initial condition gives $C = x$ or

$$\xi = \epsilon + x. \quad (1.2.23)$$

Thus we see that the infinitesimal group does determine the group!

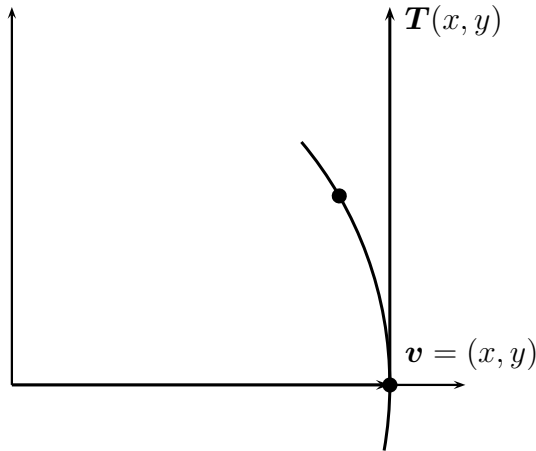
Example - Rotations - Continued. The rotation group in two dimensions is given by

$$\mathbf{G}(\epsilon, x, y) = (\cos(\epsilon)x - \sin(\epsilon)y, \sin(\epsilon)x + \cos(\epsilon)y). \quad (1.2.24)$$

The infinitesimal group is given by

$$\begin{aligned} \mathbf{T}(x, y) &= \frac{d}{d\epsilon} \mathbf{G}(\epsilon, x, y) \big|_{\epsilon=0} \\ &= \frac{d}{d\epsilon} (\cos(\epsilon)x - \sin(\epsilon)y, \sin(\epsilon)x + \cos(\epsilon)y) \big|_{\epsilon=0} \\ &= (-\sin(\epsilon)x - \cos(\epsilon)y, \cos(\epsilon)x - \sin(\epsilon)y) \big|_{\epsilon=0} \\ &= (-y, x). \end{aligned} \quad (1.2.25)$$

We can now make a sketch of the infinitesimal group which is nothing more than the direction field (with magnitude) given by $\mathbf{T}(x, y) = (-y, x)$



Infinitesimal Rotations

The initial value problem for determining $\mathbf{G}(\epsilon, x, y)$ in terms of $\mathbf{T}(x, y)$ is easier to write down if we introduce

$$(\xi, \eta) = (\xi(\epsilon), \eta(\epsilon)) = (G_1(\epsilon, x, y), G_2(\epsilon, x, y)). \quad (1.2.26)$$

Then the initial value problem becomes

$$\frac{d}{d\epsilon}(\xi, \eta) = T(\xi, \eta) = (-\eta, \xi), (\xi(0), \eta(0)) = (x, y) \quad (1.2.27)$$

or

$$\frac{d\xi}{d\epsilon} = -\eta, \frac{d\eta}{d\epsilon} = \xi, \xi(0) = x, \eta(0) = y. \quad (1.2.28)$$

This simple initial value problem can be solved in many different ways. We note that

$$\frac{d^2\xi}{d\epsilon^2} + \xi = 0 \quad (1.2.29)$$

so that

$$\xi = A \cos(\epsilon) + B \sin(\epsilon) \quad (1.2.30)$$

and then

$$\eta = -\frac{d\xi}{d\epsilon} = -B \cos(\epsilon) + A \sin(\epsilon) \quad (1.2.31)$$

The initial conditions then give

$$A = x, B = y \quad (1.2.32)$$

or

$$\xi = x \cos(\epsilon) - y \sin(\epsilon), \eta = x \sin(\epsilon) + y \cos(\epsilon). \quad (1.2.33)$$

Again we see that the infinitesimal group does determine the group.

There is an alternate method for determining the infinitesimal group that uses power series. The expansion of $\mathbf{G}(\epsilon, \mathbf{v})$ at $\epsilon = 0$ is given by

$$\begin{aligned} \mathbf{G}(\epsilon, \mathbf{v}) &= \mathbf{G}(0, \mathbf{v}) + \frac{d}{d\epsilon} \mathbf{G}(\epsilon, \mathbf{v}) \big|_{\epsilon=0} \epsilon + \frac{d^2}{d\epsilon^2} \mathbf{G}(\epsilon, \mathbf{v}) \big|_{\epsilon=0} \frac{\epsilon^2}{2} + \cdots \\ &= \mathbf{v} + \mathbf{T}(\mathbf{v})\epsilon + \cdots \end{aligned}$$

Thus $\mathbf{T}(\mathbf{v})$ is the coefficient of ϵ in the power series expansion of $\mathbf{G}(\epsilon, \mathbf{v})$.

Example - Rotations - Continued. As before

$$\begin{aligned}
G(\epsilon, x, y) &= (\cos(\epsilon)x - \sin(\epsilon)y, \sin(\epsilon)x + \cos(\epsilon)y) \\
&= \left(\left(1 - \frac{\epsilon^2}{2} + \cdots\right)x - \left(\epsilon - \frac{\epsilon^3}{6} + \cdots\right)y, \right. \\
&\quad \left. \left(\epsilon - \frac{\epsilon^3}{6} + \cdots\right)x + \left(1 - \frac{\epsilon^2}{2} + \cdots\right)y \right) \\
&= (x, y) + (-y, x)\epsilon + (-x, -y)\frac{\epsilon^2}{2} + \cdots
\end{aligned} \tag{1.2.34}$$

and consequently

$$\mathbf{T}(x, y) = (-y, x). \tag{1.2.35}$$

Here is a catalog of some elementary one dimensional examples that appear in applications.

$\xi(\epsilon) = G(\epsilon, x)$	$T(x)$	common name
$\xi(\epsilon) = x + \epsilon$	$T(x) = 1$	translations
$\xi(\epsilon) = e^\epsilon x$	$T(x) = x$	dilations
$\xi(\epsilon) = x/(1 - \epsilon x)$	$T(x) = x^2$	conformal maps

Examples

Example - Conformal. If

$$\xi(\epsilon) = x/(1 - \epsilon x), \tag{1.2.36}$$

then

$$T(x) = \frac{d}{d\epsilon} \xi(\epsilon) \big|_{\epsilon=0} = \frac{x^2}{(1 - \epsilon x)^2} \big|_{\epsilon=0} = x^2. \tag{1.2.37}$$

Alternately, the power series for $1/(1 - x)$ is given by

$$\frac{1}{1 - x} = 1 + x + O(x^2) \tag{1.2.38}$$

and consequently

$$\frac{x}{1 - \epsilon x} = x(1 + \epsilon x + O(\epsilon x)^2) = x + \epsilon x^2 + O(\epsilon^2). \tag{1.2.39}$$

As before,

$$T(x) = x^2. \tag{1.2.40}$$

On the other hand, if we know that $T(x) = x^2$, then the group $\xi(\epsilon) = G(\epsilon, x)$ satisfies the initial value problem

$$\frac{d\xi}{d\epsilon} = \xi^2, \xi(0) = x. \quad (1.2.41)$$

This initial value problem can be solved by separating variables and the result is

$$\xi(\epsilon) = \frac{x}{1 - \epsilon x}. \quad (1.2.42)$$

Exercise. Repeat the above calculation for the remainder of the previous table.

In two or more dimensions it is helpful to use column vector and matrix notation. To see how to do this we redo the rotation example.

Example - Rotation - Continued. We rewrite the rotation group in the form

$$\begin{bmatrix} \xi(\epsilon) \\ \eta(\epsilon) \end{bmatrix} = \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} \cos(\epsilon) & -\sin(\epsilon) \\ \sin(\epsilon) & \cos(\epsilon) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \quad (1.2.43)$$

The infinitesimal group is given by

$$\begin{aligned} \frac{d}{d\epsilon} \begin{bmatrix} \xi \\ \eta \end{bmatrix} \Big|_{\epsilon=0} &= \begin{bmatrix} -\sin(\epsilon) & -\cos(\epsilon) \\ -\cos(\epsilon) & -\sin(\epsilon) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Big|_{\epsilon=0} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} -y \\ x \end{bmatrix}. \end{aligned} \quad (1.2.44)$$

The differential equations for determining the rotation group in terms of the infinitesimal rotation group are then

$$\begin{aligned} \frac{d}{d\epsilon} \begin{bmatrix} \xi \\ \eta \end{bmatrix} &= \begin{bmatrix} \xi' \\ \eta' \end{bmatrix} = \begin{bmatrix} -\eta \\ \xi \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \\ \begin{bmatrix} \xi(0) \\ \eta(0) \end{bmatrix} &= \begin{bmatrix} x \\ y \end{bmatrix}. \end{aligned} \quad (1.2.45)$$

These are the same differential equations that occurred previously in this example. Note that the solution of this *constant coefficient linear* system of

differential equations can be written using *matrix exponentials* as

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = e^{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \epsilon} \begin{bmatrix} x \\ y \end{bmatrix}. \quad (1.2.46)$$

We will discuss the meaning of this notation shortly.

Let us now turn to some examples in R^n . If $\mathbf{a} = (a_1, \dots, a_n)$ is any given vector, then the translations in the direction \mathbf{a} are written

$$\mathbf{G}(\epsilon, \mathbf{v}) = \mathbf{v} + \epsilon \mathbf{a}. \quad (1.2.47)$$

Exercise. Check that this is a group of transformations with infinitesimal

$$\mathbf{T}(\mathbf{v}) = \mathbf{a}. \quad (1.2.48)$$

Make a sketch of this group and its infinitesimal!

If A is any $n \times n$ real matrix then A will generate a transformation group on R^n that is analogous to the rotation group. This analogy is based on matrix exponentials [9, Chapter 10]. The definition of the exponential of a matrix is a direct generalization of the power series definition of an exponential of a number.

Definition. If A is a given matrix, then

$$e^{\epsilon A} = \sum_{n=0}^{\infty} \epsilon^n \frac{A^n}{n!} = I + \epsilon A + \frac{\epsilon^2 A^2}{2} + \dots \quad (1.2.49)$$

where I is the identity matrix.

Note. The parameter ϵ may be omitted (choose $\epsilon = 1$) in this definition. However, we will find this parameter useful.

It is a standard result that this series converges for all A and ϵ and that the entries in the matrix

$$M(\epsilon) = \exp(\epsilon A) = e^{\epsilon A} \quad (1.2.50)$$

are analytic functions of ϵ and the entries of A . Clearly

$$e^{0A} = I. \quad (1.2.51)$$

An argument involving only rearranging summations will show that

$$e^{\epsilon_1 A} e^{\epsilon_2 A} = e^{(\epsilon_1 + \epsilon_2) A} \quad (1.2.52)$$

and then this identity implies that

$$e^{-\epsilon A} e^{\epsilon A} = I = e^{\epsilon A} e^{-\epsilon A}. \quad (1.2.53)$$

Note that

$$A e^{\epsilon A} = e^{\epsilon A} A \quad (1.2.54)$$

and that

$$\frac{d}{d\epsilon} e^{\epsilon A} = \sum_{n=1}^{\infty} \epsilon^{n-1} \frac{A^n}{(n-1)!} = A \sum_{m=0}^{\infty} \epsilon^m \frac{A^m}{m!} = A e^{\epsilon A}, \quad (1.2.55)$$

that is, the matrix function $M(\epsilon) = \exp(\epsilon A)$ is a one parameter group of matrices and is a solution of the initial value problem

$$\frac{d}{d\epsilon} M(\epsilon) = A M(\epsilon), M(0) = I. \quad (1.2.56)$$

Frequently $M(\epsilon)$ is called a fundamental solution matrix for the differential equation (1.2.56).

Exercise. Suppose that a matrix function $M(\epsilon)$ satisfies

$$M(0) = I, M(\epsilon_1) M(\epsilon_2) = M(\epsilon_1 + \epsilon_2). \quad (1.2.57)$$

Then show that

$$M(\epsilon) = e^{\epsilon A} \quad (1.2.58)$$

where

$$A = \frac{d}{d\epsilon} M(\epsilon) \big|_{\epsilon=0}. \quad (1.2.59)$$

It is now possible to generate many groups of transformations on R^n .

Proposition. If A is any real $n \times n$ matrix, and $M(\epsilon) = \exp(\epsilon A)$, then

$$\boldsymbol{\nu} = \boldsymbol{\nu}(\epsilon) = M(\epsilon) \boldsymbol{v} \quad (1.2.60)$$

is a transformation group on R^n with infinitesimal

$$\boldsymbol{T}(\boldsymbol{v}) = A \boldsymbol{v}. \quad (1.2.61)$$

Exercise. Prove this proposition.

Exercise. Show that the matrix

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \quad (1.2.62)$$

generates the group

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} e^{\epsilon a} x \\ e^{\epsilon b} y \end{bmatrix} \quad (1.2.63)$$

which is called a dilation group. Note that a similar result holds for any diagonal matrix.

One useful way to evaluate matrix exponentials is based on diagonalization results. Thus suppose that S is a matrix whose columns are linearly independent eigenvectors of A . Then

$$D = S^{-1}AS \quad (1.2.64)$$

is a diagonal matrix. The power series definition of the exponential then gives

$$e^{\epsilon A} = e^{\epsilon SDS^{-1}} = Se^{\epsilon D}S^{-1} \quad (1.2.65)$$

and, as we saw above, $\exp(\epsilon D)$ is easy to compute

$$e^{\epsilon \begin{pmatrix} \lambda_1 & 0 & \cdots \\ 0 & \lambda_2 & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}} = \begin{pmatrix} e^{\epsilon \lambda_1} & 0 & \cdots \\ 0 & e^{\epsilon \lambda_2} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}. \quad (1.2.66)$$

Exercise. Show that the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (1.2.67)$$

generates the group

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} \cosh(\epsilon) & \sinh(\epsilon) \\ \sinh(\epsilon) & \cosh(\epsilon) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \quad (1.2.68)$$

Exercise. Use the diagonalization procedure to show that

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad (1.2.69)$$

generates the rotation group. Note that D and S may be complex matrices.

1.3 Action on Functions and Lie Series.

If $\mathbf{G}(\epsilon, \mathbf{v})$ is a transformation group on R^n , then \mathbf{G} has a natural action on functions, written $G(\epsilon, f)$, and given by

$$G(\epsilon, f)(\mathbf{v}) = f(\mathbf{G}(\epsilon, \mathbf{v})) . \quad (1.3.70)$$

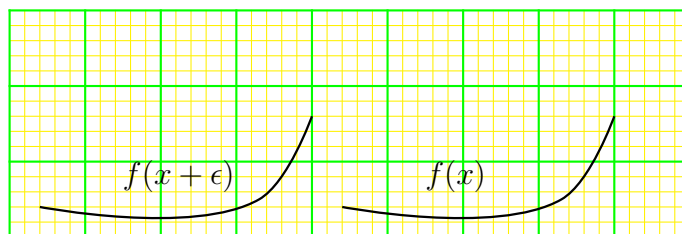
It is an abuse of notation to use the letter G to label both the transformation group and the action of the group on functions. However, these objects are so closely related that we will come to identify them and then the notation is appropriate. Note that G is not bold face in the action on functions formula, (1.3.70). Some authors put a minus sign in the definition of the action on functions

$$G(\epsilon, f)(\mathbf{v}) = f(\mathbf{G}(-\epsilon, \mathbf{v})) . \quad (1.3.71)$$

We will *not* use the minus sign.

Example - Translations. If f is a function of the real variable x , $y = f(x)$, and $G(\epsilon, x) = x + \epsilon$ is the translation group, then

$$G(\epsilon, f)(x) = f(x + \epsilon) . \quad (1.3.72)$$



Translation of Functions

Thus we see that right translation of points corresponds to left translation of functions. If we had included the minus sign in the definition of G , then right translation of points would have corresponded to right translation of functions.

We now compute the infinitesimal transformation group of $G(\epsilon, f)$.

Proposition. If $\mathbf{G}(\epsilon, \mathbf{v})$ is a one parameter transformation group on R^n with infinitesimal $\mathbf{T}(\mathbf{v})$, $f(\mathbf{v})$ is a function on R^n and $G(\epsilon, f)$ is defined by

$$G(\epsilon, f)(\mathbf{v}) = f(\mathbf{G}(\epsilon, \mathbf{v})), \quad (1.3.73)$$

then the infinitesimal $T(f)$ of $G(\epsilon, f)$ is given by

$$T(f)(\mathbf{v}) = \mathbf{T}(\mathbf{v}) \cdot \nabla f(v) \quad (1.3.74)$$

where

$$\nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right) \quad (1.3.75)$$

Proof. The infinitesimal is given by

$$T(f)(\mathbf{v}) = \frac{d}{d\epsilon} G(\epsilon, f)(\mathbf{v}) \big|_{\epsilon=0} = \frac{d}{d\epsilon} f(\mathbf{G}(\epsilon, \mathbf{v})) \big|_{\epsilon=0} . \quad (1.3.76)$$

The chain rule then gives

$$\begin{aligned} T(f)(\mathbf{v}) &= \sum_{k=1}^n \frac{\partial f}{\partial x_k}(\mathbf{G}(\epsilon, \mathbf{v})) \frac{dG_k}{d\epsilon}(\epsilon, \mathbf{v}) \big|_{\epsilon=0} \\ &= \sum_{k=1}^n \frac{\partial f}{\partial x_k}(\mathbf{v}) T_k(\mathbf{v}) \\ &= \mathbf{T}(\mathbf{v}) \cdot \nabla f(\mathbf{v}) . \end{aligned} \quad (1.3.77)$$

This result provides us with an identification of vector fields with first order partial differential operators,

$$\mathbf{T}(\mathbf{v}) < - > \mathbf{T}(\mathbf{v}) \cdot \nabla f = T(f) . \quad (1.3.78)$$

This identification is used extensively in the literature and in this manuscript.

In our work we will meet situations where we are given a vector field $\mathbf{T}(\mathbf{v})$, or equivalently a first order partial differential operator $\mathbf{T}(\mathbf{v}) \cdot \nabla$ and we will want to find the group action associated with the vector field or operator. A useful representation of group action is given by the exponential of the differential operator

$$e^{\epsilon \mathbf{T}(\mathbf{v}) \cdot \nabla} . \quad (1.3.79)$$

Such exponentials are called *Lie series*. Lie series have been studied extensively [10, 101, 188, 189]. For the convenience of the reader we now list the properties of Lie series. The proofs of the first 6 properties are not difficult and proofs of all properties are included in the cited literature. After we state the properties we will give some examples.

A Lie series is an exponential

$$e^{tL} = \sum_{k=0}^{\infty} \frac{t^k L^k}{k!} \quad (1.3.80)$$

of a first order differential operator

$$L = \mathbf{T}(\mathbf{v}) \cdot \nabla = \sum_{i=1}^n T_i(\mathbf{v}) \frac{\partial}{\partial x_i} \quad (1.3.81)$$

in the n variables $\mathbf{v} = (x_1, \dots, x_n)$. A Lie series operates on scalar functions $g(\mathbf{v})$ that are analytic in the n variables $\mathbf{v} = (x_1, \dots, x_n)$. The action of the Lie series on a function $g(\mathbf{v})$, analytic near $\mathbf{v} = 0$, is given by

$$e^{\epsilon L} g(\mathbf{v}) = \sum_{n=0}^{\infty} \frac{\epsilon^n L^n}{n!} g(\mathbf{v}) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \left(\sum_{i=1}^n f_i(\mathbf{v}) \frac{\partial}{\partial x_i} \right)^n g(\mathbf{v}) . \quad (1.3.82)$$

PROPERTIES

Assume that $f(\mathbf{v})$, $g(\mathbf{v})$ and $h(\mathbf{v})$ are analytic functions near $\mathbf{v} = 0$, that a and b are real constants, that $c(\epsilon)$ is an analytic real valued function of ϵ and that L is as in 1.81.

1) Convergence.

$$e^{\epsilon L} g(\mathbf{v}) \quad (1.3.83)$$

is a well defined analytic function of \mathbf{v} and ϵ for \mathbf{v} and ϵ small. This is a simple version of the Cauchy-Kowalewski theorem.

2) Time Derivative.

$$\frac{d}{dt} e^{c(\epsilon)L} = c'(\epsilon) L e^{c(\epsilon)L} = e^{c(\epsilon)L} c'(\epsilon) L \quad (1.3.84)$$

3) Linearity.

$$e^{\epsilon L} (ag + bh) = ae^{\epsilon L} g + be^{\epsilon L} h \quad (1.3.85)$$

4) Product Preservation.

$$e^{\epsilon L} (gh) = (e^{\epsilon L} g)(e^{\epsilon L} h) \quad (1.3.86)$$

Lie Series act on vector valued functions $\mathbf{f}(\mathbf{v}) = (f_1(\mathbf{v}), \dots)$, by acting on each component,

$$e^{\epsilon L} \mathbf{f}(\mathbf{v}) = (e^{\epsilon L} f_1(\mathbf{v}), \dots) . \quad (1.3.87)$$

5) **Composition.**

$$e^{\epsilon L} g(\mathbf{v}) = g(e^{\epsilon L} \mathbf{v}), \quad (1.3.88)$$

6) **Differential Equation Property.**

If

$$\mathbf{v}(\epsilon) = e^{\epsilon L} \mathbf{v} \quad (1.3.89)$$

then

$$\mathbf{v}'(\epsilon) = f(\mathbf{v}(\epsilon)), \mathbf{v}(0) = \mathbf{v} . \quad (1.3.90)$$

We now suppose that P is another first order differential operator and define successive commutators by:

$$\begin{aligned} [L, \cdot]^0 P &= P \\ [L, \cdot]^1 P &= LP - PL \\ [L, \cdot]^n P &= [L, \cdot]^{n-1} [L, P], n \geq 1 \end{aligned} \quad (1.3.91)$$

7) **Similarity.**

$$e^{\epsilon L} P e^{-\epsilon L} = e^{\epsilon [L, \cdot]} P = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} [L, \cdot]^k P \quad (1.3.92)$$

8) **Function Multiplier.**

$$e^{\epsilon L} g e^{-\epsilon L} h = (e^{\epsilon L} g) h \quad (1.3.93)$$

Although we do not yet have a use for the next property we list it because it is very useful in other contexts, has important Lie algebraic applications and is certainly the most interesting elementary fact about Lie series.

9) **Noncommuting Exponential Identities.**

$$\begin{aligned} e^{\epsilon(L+P)} &= e^{\epsilon L} e^{\epsilon P} e^{\epsilon^2 L_2} e^{\epsilon^3 L_3} e^{\epsilon^4 L_4} e^{\epsilon^5 L_5} \dots \\ &= \dots e^{\epsilon^5 L_5} e^{-\epsilon^4 L_4} e^{\epsilon^3 L_3} e^{-\epsilon^2 L_2} e^{\epsilon P} e^{\epsilon L} \\ e^{\epsilon L} e^{\epsilon P} &= e^{\epsilon L + \epsilon P + \epsilon^2 W_2 + \epsilon^3 W_3 + \dots} \end{aligned} \quad (1.3.94)$$

where

$$L_2 = -\frac{1}{2}[L, P] \quad , \quad W_2 = \frac{1}{2}[L, P], \quad (1.3.95)$$

and so forth. Here each L_k and W_k are k -fold commutators of L and P .

Example - Translations - Continued.

The infinitesimal action of translations is given by the differential operator

$$L = \frac{d}{dx} \quad (1.3.96)$$

and consequently the one parameter group of translations should be given by

$$e^{\epsilon \frac{d}{dx}} . \quad (1.3.97)$$

The Composition Property gives

$$e^{\epsilon \frac{d}{dx}} f(x) = f(e^{\epsilon \frac{d}{dx}} x) \quad (1.3.98)$$

so we need only compute

$$e^{\epsilon \frac{d}{dx}} x = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \left(\frac{d}{dx} \right)^k x = x + \epsilon \quad (1.3.99)$$

as was desired.

Exercise - Dilations - Continued.

As in the above example, show that

$$e^{\epsilon x \frac{d}{dx}} f(x) = f(e^{\epsilon} x) . \quad (1.3.100)$$

Exercise - Conformal - Continued.

Show that

$$e^{\epsilon x^2 \frac{d}{dx}} f(x) = f\left(\frac{x}{1 - \epsilon x}\right) . \quad (1.3.101)$$

Example - Rotations - Continued.

The vector field that is the infinitesimal of the rotation group is

$$\mathbf{T}(x, y) = (-y, x) \quad (1.3.102)$$

so the infinitesimal action on functions is given by

$$L = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} , \quad (1.3.103)$$

that is,

$$(Lf)(x, y) = -y \frac{\partial f}{\partial x}(x, y) + x \frac{\partial f}{\partial y}(x, y) . \quad (1.3.104)$$

Consequently the one parameter group of rotations is given by

$$e^{\epsilon L} f(x, y) = f(e^{\epsilon L} x, e^{\epsilon L} y) \quad (1.3.105)$$

so we need only compute

$$e^{\epsilon L} x, \quad e^{\epsilon L} y . \quad (1.3.106)$$

The power series gives

$$\begin{aligned} e^{\epsilon L} x &= \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \left(-y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right)^k x \\ &= x + \epsilon(-y) + \frac{\epsilon^2}{2}(-x) + \frac{\epsilon^3}{3!}y + \cdots \\ &= \cos(\epsilon)x - \sin(\epsilon)y . \end{aligned} \quad (1.3.107)$$

Similarly

$$e^{\epsilon L} y = \sin(\epsilon)x + \cos(\epsilon)y . \quad (1.3.108)$$

Exercise. If A is any $n \times n$ matrix, $A = (a_{ij})$ and

$$L = \mathbf{v} A \nabla = \sum_{i,j=1}^n x_i a_{ij} \frac{\partial}{\partial x_j} , \quad (1.3.109)$$

then show that

$$e^{\epsilon L} f(\mathbf{v}) = f(e^{\epsilon A} \mathbf{v}) .$$

In this sense Lie series generalize the matrix exponential.

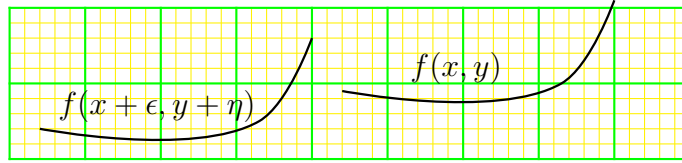
It is the author's opinion that the Lie series formalism is an invaluable tool for understanding transformation groups. However, the power series method of evaluating Lie series is not very powerful. Thus, when trying to evaluate a complicated Lie series it is better to convert the problem to a system of ordinary differential equations using the Differential Equation Property and then apply classical ordinary differential equation techniques to the problem.

Exercise. Use the Differential Equation Property to redo some of the previous examples.

1.4 Action on Curves and Surfaces.

This section will bring us substantially closer to the purpose of this manuscript; the study of differential equations. The solutions of a differential equation or system of differential equations can be interpreted as a curve, surface or in general, a hyper-surface in some Euclidean space. We want to know how transformation groups on the Euclidean space act on such curves and surfaces. To do this we need to expand the notation of the previous sections.

Before we study the general case let us look at the situation in R^2 .



Motion of a Curve in R^2

Now let

$$\mathbf{G}(\epsilon, x, y) = (\xi(\epsilon, x, y), \eta(\epsilon, x, y)) \quad (1.4.110)$$

be a group of transformations with infinitesimal

$$\mathbf{T}(x, y) = (r(x, y), s(x, y)) . \quad (1.4.111)$$

If $y = f(x)$ is a curve, then $y = f(\epsilon, x)$ is to be a curve whose graph is the image of the graph of $y = f(x)$ under the action of the group element $G(\epsilon, x, y)$. Under the action of $G(-\epsilon, x, y)$ the points $(x, f(\epsilon, x))$ go into the points $(\xi(-\epsilon, x, f(\epsilon, x)), \eta(-\epsilon, x, f(\epsilon, x)))$ which must lie in the graph of $y = f(x)$, that is,

$$\eta(-\epsilon, x, f(\epsilon, x)) = f(\xi(-\epsilon, x, f(\epsilon, x))) \quad (1.4.112)$$

If $f(\epsilon, x)$ is replaced by y , then (1.4.112) becomes

$$\eta(-\epsilon, x, y) = f(\xi(-\epsilon, x, y)) \quad (1.4.113)$$

which can be solved for $y = f(\epsilon, x)$. Thus (1.4.112) which gives an implicit definition of $f(\epsilon, x)$. In some elementary cases (1.4.112) can be solved for $f(\epsilon, x)$ but, in general, this cannot be done. However, it is possible to compute the *explicit* infinitesimal action!

The computation of the infinitesimal action of the group on curves requires some formulas. Recall that

$$\begin{aligned}\xi(0, x, y) &= x, & \eta(0, x, y) &= y, \\ \xi_\epsilon(0, x, y) &= r(x, y) & \eta_\epsilon(0, x, y) &= s(x, y),\end{aligned}$$

and consequently

$$\xi_y(0, x, y) = 0, \quad \eta_y(0, x, y) = 1. \quad (1.4.114)$$

Differentiate (1.4.112) with respects to ϵ and set $\epsilon = 0$. Then

$$-s(x, f(x)) + f_\epsilon(0, x) = -f'(x)r(x, f(x)). \quad (1.4.115)$$

Solving for f_ϵ gives

$$f_\epsilon(0, x) = -r(x, f(x))\frac{df(x)}{dx} + s(x, f(x)) \quad (1.4.116)$$

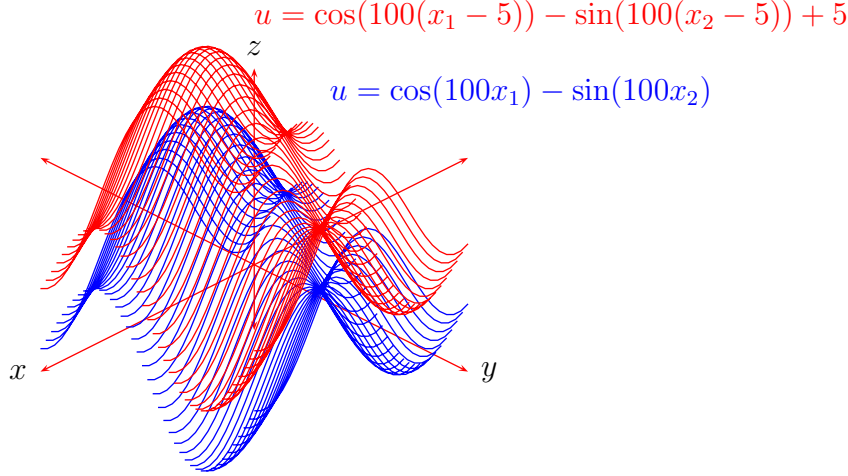
which is the infinitesimal action. Set

$$L(f)(x) = -r(x, f(x))\frac{df(x)}{dx} + s(x, f(x)) \quad (1.4.117)$$

and note that L is a nonlinear differential operator that gives the infinitesimal group action.

To study the general case let n and m be two positive integers. We will now consider transformation groups on R^{n+m} where the points in R^{n+m} are labeled with two vectors $\mathbf{u} = (u_1, \dots, u_m)$ and $\mathbf{x} = (x_1, \dots, x_n)$. The variables \mathbf{u} will be considered *dependent* while the variables \mathbf{x} will be considered *independent*, that is, we will be considering surfaces (hyper-surfaces) of the form

$$\mathbf{u} = \mathbf{f}(\mathbf{x}). \quad (1.4.118)$$



Motion of a Surface in R^{n+m}

A transformation group on R^{n+m} will be written $\mathbf{G}(\epsilon, \mathbf{x}, \mathbf{u})$. Geometrically it is clear that the transformation group will move the surface given by the function \mathbf{f} into a new surface given by a function $\mathbf{f}(\epsilon, \mathbf{x})$. Actually there may be exceptional points where the new surface is vertical, but such singularities will not give us any difficulty. First we write

$$\mathbf{G}(\epsilon, \mathbf{x}, \mathbf{u}) = (\mathbf{G}_1(\epsilon, \mathbf{x}, \mathbf{u}), \mathbf{G}_2(\epsilon, \mathbf{x}, \mathbf{u})) \quad (1.4.119)$$

where \mathbf{G}_1 gives the \mathbf{x} components of \mathbf{G} and \mathbf{G}_2 gives the \mathbf{u} components. We also write the infinitesimal of \mathbf{G} in a similar way;

$$\mathbf{T}(\mathbf{x}, \mathbf{u}) = (\mathbf{T}_1(\mathbf{x}, \mathbf{u}), \mathbf{T}_2(\mathbf{x}, \mathbf{u})) . \quad (1.4.120)$$

The differential operator corresponding to \mathbf{T} is then

$$\mathbf{T}_1(\mathbf{x}, \mathbf{u}) \cdot \nabla_{\mathbf{x}} + \mathbf{T}_2(\mathbf{x}, \mathbf{u}) \nabla_{\mathbf{u}} \quad (1.4.121)$$

where

$$\nabla_{\mathbf{x}} = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right), \quad \nabla_{\mathbf{u}} = \left(\frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_m} \right) . \quad (1.4.122)$$

Now if the point (\mathbf{x}, \mathbf{u}) is on the surface given by $\mathbf{u} = \mathbf{f}(\epsilon, \mathbf{x})$ then the point $(\mathbf{G}_1(-\epsilon, \mathbf{x}, \mathbf{f}(\mathbf{x})), \mathbf{G}_2(-\epsilon, \mathbf{x}, \mathbf{f}(\mathbf{x})))$ is on the surface given by $\mathbf{u} = \mathbf{f}(\mathbf{x})$, that is,

$$\mathbf{G}_2(-\epsilon, \mathbf{x}, \mathbf{u}) = \mathbf{f}(\mathbf{G}_1(-\epsilon, \mathbf{x}, \mathbf{u})) . \quad (1.4.123)$$

If this equation is solved for \mathbf{u} as a function of ϵ and \mathbf{x} , then the result defines $\mathbf{u} = \mathbf{f}(\epsilon, \mathbf{x})$. The implicit function theorem guarantees that such a solution exists for ϵ sufficiently small and \mathbf{x} in some small set. Thus $\mathbf{u} = \mathbf{f}(\epsilon, \mathbf{x})$ is implicitly defined by

$$\mathbf{G}_2(-\epsilon, \mathbf{x}, \mathbf{f}(\epsilon, \mathbf{x})) = \mathbf{f}(\mathbf{G}_1(-\epsilon, \mathbf{x}, \mathbf{f}(\epsilon, \mathbf{x}))) . \quad (1.4.124)$$

Also, the group action on the function $\mathbf{u} = \mathbf{f}(\mathbf{x})$ is then written

$$\mathbf{G}(\epsilon, \mathbf{f})(\mathbf{x}) = \mathbf{f}(\epsilon, \mathbf{x}) . \quad (1.4.125)$$

The infinitesimal action on surfaces is

$$\mathbf{L}(\mathbf{f})(\mathbf{x}) = \frac{d}{d\epsilon} \mathbf{G}(\epsilon, \mathbf{f})(\mathbf{x}) \big|_{\epsilon=0} = \frac{d}{d\epsilon} \mathbf{f}(\epsilon, \mathbf{x}) \big|_{\epsilon=0} . \quad (1.4.126)$$

This is computed by differentiating (1.4.124) implicitly with respects to ϵ . The derivative of the left hand side of (1.4.124) at $\epsilon = 0$ is

$$\begin{aligned} & \frac{d}{d\epsilon} \mathbf{G}_2(-\epsilon, \mathbf{x}, \mathbf{f}(\epsilon, \mathbf{x})) \big|_{\epsilon=0} = \\ & -\mathbf{T}_2(\mathbf{G}_2(-\epsilon, \mathbf{x}, \mathbf{f}(\epsilon, \mathbf{x}))) \big|_{\epsilon=0} + \nabla_{\mathbf{u}} \mathbf{G}_2(-\epsilon, \mathbf{x}, \mathbf{f}(\epsilon, \mathbf{x})) \cdot \frac{d}{d\epsilon} \mathbf{f}(\epsilon, \mathbf{x}) \big|_{\epsilon=0} . \end{aligned} \quad (1.4.127)$$

Recall that $\mathbf{G}_2(0, \mathbf{x}, \mathbf{u}) = \mathbf{u}$ so that $\nabla_{\mathbf{u}} \mathbf{G}_2(0, \mathbf{x}, \mathbf{u})$ is an identity matrix. Thus the right-hand side of (1.4.127) becomes

$$-\mathbf{T}_2(\mathbf{x}, \mathbf{f}(\mathbf{x})) + \mathbf{L}(\mathbf{f})(\mathbf{x}) . \quad (1.4.128)$$

The derivative of the right-hand side of (1.4.124) at $\epsilon = 0$ is

$$\begin{aligned} & \nabla_{\mathbf{x}} \mathbf{f}(\mathbf{G}_1(-\epsilon, \mathbf{x}, \mathbf{f}(\epsilon, \mathbf{x}))) = \\ & \{-\mathbf{T}_1(\mathbf{G}_1(-\epsilon, \mathbf{x}, \mathbf{f}(\epsilon, \mathbf{x}))) + \nabla_{\mathbf{u}} \mathbf{G}_1(-\epsilon, \mathbf{x}, \mathbf{f}(\epsilon, \mathbf{x})) \frac{d}{d\epsilon} \mathbf{f}(\epsilon, \mathbf{x})\} \big|_{\epsilon=0} . \end{aligned} \quad (1.4.129)$$

However, $\mathbf{G}_1(0, \mathbf{x}, \mathbf{u}) = \mathbf{x}$ and $\nabla_{\mathbf{u}} \mathbf{x} = 0$ so (1.4.129) becomes

$$-\mathbf{T}_1(\mathbf{x}, \mathbf{f}(\mathbf{x})) \cdot \nabla_{\mathbf{x}} \mathbf{f}(\mathbf{x}) . \quad (1.4.130)$$

Combining (1.4.128) and (1.4.130) gives

$$-T_2(\mathbf{x}, \mathbf{f}(\mathbf{x})) + \mathbf{L}(\mathbf{f})(\mathbf{x}) = -T_1(\mathbf{x}, \mathbf{f}(\mathbf{x})) \cdot \nabla_{\mathbf{x}} \mathbf{f}(\mathbf{x}) \quad (1.4.131)$$

or

$$\mathbf{L}(\mathbf{f})(\mathbf{x}) = -T_1(\mathbf{x}, \mathbf{f}(\mathbf{x})) \nabla_{\mathbf{x}} \mathbf{f}(\mathbf{x}) + T_2(\mathbf{x}, \mathbf{f}(\mathbf{x})) . \quad (1.4.132)$$

We summarize this.

Proposition.

If $(T_1(\mathbf{x}, \mathbf{u}), T_2(\mathbf{x}, \mathbf{u}))$ is a vector field on R^{n+m} , the action of the group generated by the vector field on functions $\mathbf{u} = \mathbf{f}(\mathbf{x})$ has an infinitesimal given by

$$\mathbf{L}(\mathbf{f})(\mathbf{x}) = -T_1(\mathbf{x}, \mathbf{f}(\mathbf{x})) \cdot \nabla_{\mathbf{x}} \mathbf{f}(\mathbf{x}) + T_2(\mathbf{x}, \mathbf{f}(\mathbf{x})) . \quad (1.4.133)$$

Note that \mathbf{L} is a *nonlinear* (quasi-linear) first order differential operator. This correspondence is easy to remember because

$$(T_1 \cdot \nabla_{\mathbf{x}} + T_2 \cdot \nabla_{\mathbf{u}})(\mathbf{u} - \mathbf{f}(\mathbf{x})) = T_2 - T_1 \cdot \nabla_{\mathbf{x}} \mathbf{f}(\mathbf{x}) . \quad (1.4.134)$$

1.5 Invariants and Canonical Coordinates.

An important technique that we will use to study problems involves making a change of coordinates so that some group becomes particularly simple. Such transformations are found by computing functions that are invariant under the group action. Once the coordinate transformation is found, then one approach to studying the given problem is to transform everything to the new coordinates. The problem of transforming differential equations and operators to new coordinate systems which involve changes in both the dependent and independent variables is another example of a straight forward but tedious algebraic procedure. Consequently, we have provided a symbol manipulation program to do such computations. We return to the notation of the first two section to begin our discussion.

A function f mapping R^n into R is invariant under G if it satisfies

$$f(G(\epsilon, \mathbf{v})) = f(\mathbf{v}) , \quad (1.5.135)$$

that is,

$$G(\epsilon, f)(\mathbf{x}) = f(\mathbf{v}) \quad (1.5.136)$$

for all ϵ . If we differentiate this equation with respects to ϵ and set $\epsilon = 0$, we obtain

$$\mathbf{L}f(\mathbf{v}) = 0 \quad (1.5.137)$$

where \mathbf{L} is the infinitesimal of G . Conversely, if $Lf(\mathbf{v}) = 0$, then

$$G(\epsilon, f)(\mathbf{v}) = e^{\epsilon L}f(\mathbf{v}) = f(e^{\epsilon L}\mathbf{v}) = e^{\epsilon L}f(\mathbf{v}) = f(\mathbf{v}) \quad (1.5.138)$$

so that (1.5.137) and (1.5.136) are equivalent.

In general a group will have more than one invariant. If f_i , $1 \leq i \leq m$, are invariants, then they are said to be independent if the Jacobian matrix

$$\left(\frac{\partial f_i}{\partial x_j} \right) \quad 1 \leq i \leq m, \quad 1 \leq j \leq n \quad (1.5.139)$$

has maximal rank. Because the rank of the matrix (1.5.139) must be less than or equal to n , a transformation group on R^n can have no more than n independent invariants. If a group has n independent invariants, then set

$$\xi_i = f_i(\mathbf{v}) \quad (1.5.140)$$

Because the invariants are independent, the ξ_i will serve as coordinates on R^n . If $g(\mathbf{v})$ is any function, $\boldsymbol{\nu} = (\xi_1, \dots, \xi_n)$, and $\tilde{g}(\boldsymbol{\nu}) = g(\mathbf{v})$, then

$$e^{\epsilon L}g(\mathbf{v}) = e^{\epsilon L}\tilde{g}(\boldsymbol{\nu}) = \tilde{g}(e^{\epsilon L}\boldsymbol{\nu}) = \tilde{g}(\boldsymbol{\nu}) = g(\mathbf{v}) . \quad (1.5.141)$$

Thus $e^{\epsilon L}$ is the identity group. Consequently any nontrivial one parameter group on R^n can have at most $n - 1$ independent invariants.

We can now prove the following useful and elementary theorem.

Theorem. Any nontrivial one parameter transformation group on R^n has exactly $n - 1$ independent invariant functions.

Proof. We already know that the group can have at most $n - 1$ invariants. Also $f(\mathbf{v})$ is an invariant if and only if $Lf(\mathbf{v}) = 0$. This is just a first order linear partial differential equation for $f(\mathbf{v})$. The standard approach to solving this type of problem is the method of characteristics [6]. Thus, if

$$L = \mathbf{T} \cdot \boldsymbol{\nabla} = \sum T_i(\mathbf{v}) \frac{\partial}{\partial x_i} , \quad (1.5.142)$$

then the characteristic equations are

$$\frac{dx_1}{T_1(\mathbf{v})} = \frac{dx_2}{T_2(\mathbf{v})} = \dots = \frac{dx_n}{T_n(\mathbf{v})} . \quad (1.5.143)$$

If we set all these equations equal to $d\epsilon$, then we see this is just the system of equations given by the Differential Equation property listed in Section 1.3. It is well known from the theory of first order partial differential equations that (1.5.143) has $n - 1$ integrals which is just another way of saying invariants of the group. This can be seen by solving (1.5.143) for the n function $x_i = f_i(\epsilon)$, using one of the equations to eliminate ϵ , and then noting that the remaining $n - 1$ functions are invariants (or integrals). Of course these integrals are the usual integrals of the autonomous system (1.5.143).

Example - Rotations - Continued. If

$$L = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} , \quad (1.5.144)$$

then the characteristic equations are

$$\frac{dx}{-y} = \frac{dy}{x} . \quad (1.5.145)$$

Thus

$$x \, dx + y \, dy = 0 \quad (1.5.146)$$

and an integration gives

$$\frac{x^2}{2} + \frac{y^2}{2} = c . \quad (1.5.147)$$

Clearly, the function

$$f(x, y) = x^2 + y^2 \quad (1.5.148)$$

is an invariant of the rotation group.

Exercise - Dilations - Continued. The infinitesimal dilation group is

$$L = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} . \quad (1.5.149)$$

Show that $f(x, y) = y/x$ is an invariant of this group.

The next item that we will work on is the problem of giving a canonical representation of every one parameter transformation group. What we will show is that under an appropriate change of coordinates, every one parameter transformation group is isomorphic to a group of translations. As in many other branches of mathematics, this canonical representation will allow us to easily predict the outcome of many computations. It is the author's opinion

that it is hard to over-rate the usefulness of such canonical representation theorems.

Theorem. If $\mathbf{G}(\epsilon, \mathbf{v})$ is a transformation group on R^n , then there exists a change of coordinates $\boldsymbol{\nu} = \mathbf{C}(\mathbf{v})$ such that in the new coordinates $\mathbf{G}(\epsilon, \mathbf{v})$ becomes

$$\mathbf{G}(\epsilon, \boldsymbol{\nu}) = (\xi_1 + \epsilon, \xi_2, \dots, \xi_n) \quad (1.5.150)$$

Proof. If L is the infinitesimal operator of \mathbf{G} , then the theory of first order partial differential says that it is possible to choose $\xi_1 = \xi_1(\mathbf{v})$ such that

$$L\xi_1 = 1. \quad (1.5.151)$$

Next, choose $\xi_i = \xi_i(\mathbf{v})$ to be any $n - 1$ independent invariants of $G(\epsilon, \boldsymbol{\nu})$. Set $\mathbf{C}(\mathbf{v}) = (\xi_1(\mathbf{v}), \dots, \xi_n(\mathbf{v}))$. If the Jacobian of this transformation is zero, then the vectors $\nabla \xi_i = (\partial \xi_i / \partial \xi_1, \dots, \partial \xi_i / \partial \xi_n)$, $i = 1, \dots, n$ must be linearly dependent. It was assumed that ξ_2, \dots, ξ_n are linearly independent; consequently

$$\nabla \xi_1 = \sum_{i=2}^n \alpha_i \nabla \xi_i \quad (1.5.152)$$

for some scalars α_i . Now

$$L\xi_1 = T(\mathbf{v}) \cdot \nabla \xi_1 = \sum_{i=1}^n \alpha_i T(v) \cdot \nabla \xi_i = \sum_{i=1}^n \alpha_i L(\xi_i) = 0. \quad (1.5.153)$$

However, it was assumed that $L(\xi_1) = 1$ so the assumption of linear dependence lead to a contradiction. The linear independence implies that the Jacobian of the transformation is nonzero.

We have by definition

$$G(\epsilon, \boldsymbol{\nu}) = e^{\epsilon L} \boldsymbol{\nu} = (e^{\epsilon L} \xi_1, \dots, e^{\epsilon L} \xi_n) = (\xi_1 + \epsilon, \xi_2, \dots, \xi_n). \quad (1.5.154)$$

This canonical representation theorem immediately gives a well known canonical representation for first order differential operators.

Theorem. If L is a first order linear differential operator

$$L = \mathbf{T} \cdot \nabla, \quad T(v) = (T_1(\mathbf{v}), \dots, T_n(\mathbf{v})), \quad (1.5.155)$$

then there exists a change of coordinates $\boldsymbol{\nu} = \boldsymbol{C}(\boldsymbol{v})$ such that L transforms into \tilde{L} and

$$\tilde{L} = \frac{\partial}{\partial \xi_1} . \quad (1.5.156)$$

Proof. Under the change of coordinates given in the previous theorem, L goes into the infinitesimal generator of the translation group which is just $\partial/\partial \xi_1$.

Example - Rotations. Previously we showed that

$$x^2 + y^2 = c \quad (1.5.157)$$

is an invariant of the rotation group. Consequently all invariants are given by

$$g(x, y) = G(x^2 + y^2) \quad (1.5.158)$$

for any function G of one variable. Next we need to solve

$$Lf = 1 . \quad (1.5.159)$$

We guess that a particular solution of this equation is

$$f = \arctan \left(\frac{y}{x} \right) . \quad (1.5.160)$$

Consequently the general solution is given by

$$f(x, y) = F(x^2 + y^2) + \arctan \left(\frac{y}{x} \right) \quad (1.5.161)$$

for any function F of one variable.

Thus any change of variables

$$\xi = F(x^2 + y^2) + \arctan \left(\frac{y}{x} \right) , \quad \eta = G(x^2 + y^2) \quad (1.5.162)$$

with nonzero Jacobian will give us a set of canonical coordinates. In fact the Jacobian of such a transformation is

$$J = -2G' \quad (1.5.163)$$

so that if $G' = -1/2$ the transformation will have unit Jacobian. If the coordinates are to be orthogonal, it is easy to check that $F_{prime} = 0$ independent of the choice of G . Thus under these two constraints we have

$$\xi = c_1 + \arctan\left(\frac{y}{x}\right) , \quad \eta = c_2 - \frac{(x^2 + y^2)}{2} \quad (1.5.164)$$

which bears an obvious relationship to polar coordinates and is essentially the action-angle coordinates for the harmonic oscillator given in Hamiltonian mechanics.

Exercise - Dilations. Recall that the infinitesimal group is given by

$$L = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \quad (1.5.165)$$

Show that canonical coordinates are given by

$$\xi = F\left(\frac{y}{x}\right) + \ln |x| , \quad \eta = G\left(\frac{y}{x}\right) . \quad (1.5.166)$$

An interesting special transformation is

$$\xi = \ln \sqrt{x^2 + y^2} , \quad \eta = \arctan\left(\frac{y}{x}\right) . \quad (1.5.167)$$

Exercise. Find canonical coordinates for the operator

$$x \frac{d}{dx} , \quad x^2 \frac{d}{dx} . \quad (1.5.168)$$

Now that we know how to find canonical coordinates, we need to know how to transform differential equations to the new coordinates. Before we turn to the general case let us do the two variable problem. Thus assume that we have an infinitesimal group of the form

$$L = a(x, y) \frac{\partial}{\partial x} + b(x, y) \frac{\partial}{\partial y} \quad (1.5.169)$$

and that we have found two functions $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$ such that

$$L(\xi) = 1 , \quad L(\eta) = 0 . \quad (1.5.170)$$

The transformation that we will use is

$$\xi = \xi(x, y) , \quad \eta = \eta(x, y) . \quad (1.5.171)$$

Our theory shows that this transformation is invertible and we write the inverse as

$$x = x(\xi, \eta) , \quad y = y(\xi, \eta) . \quad (1.5.172)$$

Let us first verify that L is transformed into translation. Let $f(x, y)$ be any function and $\tilde{f}(\xi, \eta)$ be defined so that

$$\tilde{f}(\xi(x, y), \eta(x, y)) = f(x, y) . \quad (1.5.173)$$

Then the chain rule gives

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial \tilde{f}}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial \tilde{f}}{\partial \eta} \frac{\partial \eta}{\partial x} , \\ \frac{\partial f}{\partial y} &= \frac{\partial \tilde{f}}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial \tilde{f}}{\partial \eta} \frac{\partial \eta}{\partial y} \end{aligned} \quad (1.5.174)$$

and consequently

$$\begin{aligned} L(f) &= a \frac{\partial f}{\partial x} + b \frac{\partial f}{\partial y} \\ &= \left(a \frac{\partial \xi}{\partial x} + b \frac{\partial \xi}{\partial y} \right) \frac{\partial \tilde{f}}{\partial \xi} + \left(a \frac{\partial \eta}{\partial x} + b \frac{\partial \eta}{\partial y} \right) \frac{\partial \tilde{f}}{\partial \eta} \\ &= \frac{\partial \tilde{f}}{\partial \xi} = \tilde{L}(f) \end{aligned} \quad (1.5.175)$$

as was desired.

Now let us consider what happens to a curve $y = f(x)$ in the canonical coordinates. First, the curve goes into a curve $\eta = g(\xi)$ determined by solving the equation

$$y(\xi, \eta) = f(x(\xi, \eta)) \quad (1.5.176)$$

for η as a function of ξ , that is, g satisfies

$$y(\xi, g(\xi)) = f(x(\xi, g(\xi))) . \quad (1.5.177)$$

Now compute what happens to the slope of the curve $y = f(x)$. The use of differentials makes this computation transparent. Thus

$$\begin{aligned} d\xi &= \frac{\partial \xi}{\partial x} dx + \frac{\partial \xi}{\partial y} dy = \xi_x dx + \xi_y dy, \\ d\eta &= \frac{\partial \eta}{\partial x} dx + \frac{\partial \eta}{\partial y} dy = \eta_x dx + \eta_y dy \end{aligned} \quad (1.5.178)$$

and consequently

$$\eta' = \frac{d\eta}{d\xi} = \frac{\eta_x dx + \eta_y dy}{\xi_x dx + \xi_y dy} = \frac{\eta_x + \eta_y y'}{\xi_x + \xi_y y'}. \quad (1.5.179)$$

The quantities η_x, η_y, ξ_x and ξ_y still depend on x and y so the inverse transformation should be used to remove x and y in favor of ξ, η . Solving (1.5.179) for y' gives

$$y' = -\frac{\eta_x - \xi_x \eta'}{\eta_y - \xi_y \eta'}. \quad (1.5.180)$$

The action of the infinitesimal group (1.5.169) on the curve $y = f(x)$ is given by the operator

$$\begin{aligned} S(f) &= -af' + b = -ay' + b = \\ &= \frac{a(\eta_x - \xi_x \eta') + b(\eta_y - \xi_y \eta')}{\eta_y - \xi_y \eta'} \\ &= \frac{(a\eta_x + b\eta_y) - (a\xi_x + b\xi_y)\eta'}{\eta_y - \xi_y \eta'} \\ &= \frac{\eta'}{\xi_y \eta' - \eta_y}. \end{aligned} \quad (1.5.181)$$

This result seems a bit surprising. To see why we say this we summarize our results to this point. We started with an operator

$$L = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y}. \quad (1.5.182)$$

The action of this operator on curves is given by $y = f(x)$

$$S(f) = -af' + b \quad (1.5.183)$$

These two operators transform into

$$\tilde{L} = \frac{\partial}{\partial \xi} , \quad \tilde{S}(\tilde{f}) = \frac{\tilde{f}'}{\xi_y \tilde{f}' - \eta_y} . \quad (1.5.184)$$

It seems natural to have expected that $\tilde{S}\tilde{f} = \tilde{L}\tilde{f}$. However, for this to happen the transformation must be particularly simple. What is needed is

$$\xi_y = 0 , \quad \eta_y = 1 . \quad (1.5.185)$$

The original equations (1.5.170) for ξ and η then become

$$a\xi_x = 1 , \quad a\eta_x - b = 0 . \quad (1.5.186)$$

If the transformation is to have nonzero Jacobean then it must be the case that

$$\xi_x \neq 0 . \quad (1.5.187)$$

Differentiating (1.5.186) gives $a_y \xi_x = 0$, that is $a_y = 0$. Again differentiating (1.5.186) gives $b_y = 0$. For such a coordinate system

$$\xi = \xi(x) , \quad \eta = -\mu(x)y + \nu(x) \quad (1.5.188)$$

for some functions $\xi(x)$, $\mu(x)$, $\nu(x)$.

Let us now turn to the general case of transformations on R^{n+m} acting on surfaces $\mathbf{u} = \mathbf{f}(\mathbf{x})$ where $\mathbf{u} = (u_1, \dots, u_m)$ and $\mathbf{x} = (x_1, \dots, x_n)$. As before, the infinitesimal transformation group will be written

$$L = \mathbf{T}_1(\mathbf{x}, \mathbf{u}) \cdot \nabla_{\mathbf{x}} + \mathbf{T}_2(\mathbf{x}, \mathbf{u}) \cdot \nabla_{\mathbf{u}} . \quad (1.5.189)$$

Assume that we have found functions $\xi_i(\mathbf{x}, \mathbf{u})$, $1 \leq i \leq n$, $\eta_i(\mathbf{x}, \mathbf{u})$, $1 \leq i \leq m$ such that

$$\begin{aligned} L\xi_1 &= 1 , \\ L\xi_i &= 0 , \quad 2 \leq i \leq n , \\ L\eta_i &= 0 , \quad 1 \leq i \leq m . \end{aligned} \quad (1.5.190)$$

As before, the transformation

$$\boldsymbol{\xi} = (\xi_1(\mathbf{x}, \mathbf{u}), \dots, \xi_n(\mathbf{x}, \mathbf{u})) , \quad \boldsymbol{\eta} = (\eta_1(\mathbf{x}, \mathbf{u}), \dots, \eta_m(\mathbf{x}, \mathbf{u})) , \quad (1.5.191)$$

has nonzero Jacobean. Under the transformation, L is transformed into $\tilde{L} = \partial/\partial\xi_1$ as was shown in the previously.

What does the infinitesimal action on surfaces

$$S(\mathbf{u})(\mathbf{x}) = -T_1(\mathbf{x}, \mathbf{u})\nabla_x U + T_2(\mathbf{x}, \mathbf{u}) \quad (1.5.192)$$

transform into? Again, the use of differentials make the computations simple. The chain rule gives

$$d\xi = \nabla_x \xi d\mathbf{x} + \nabla_u \xi d\mathbf{u} , \quad d\eta = \nabla_x \eta d\mathbf{x} + \nabla_n \eta d\mathbf{u} . \quad (1.5.193)$$

We assume that $\mathbf{u} = \mathbf{f}(\mathbf{x})$ is transformed into $\eta = g(\xi)$. The differentials of these equations give

$$d\mathbf{u} = \nabla_x \mathbf{f} d\mathbf{x} , \quad d\eta = \nabla_\xi g d\xi . \quad (1.5.194)$$

Combining (1.5.193) and (1.5.194) gives

$$d\eta = \nabla_\xi g d\xi = \nabla_\xi g \cdot (\nabla_x \xi + \nabla_u \xi \nabla_x \mathbf{f}) d\mathbf{x} = \nabla_x \eta + \nabla_u \xi \nabla_x \mathbf{f} d\mathbf{x} . \quad (1.5.195)$$

Consequently

$$\nabla_\xi g \nabla_x \xi + \nabla_\xi g \nabla_u \xi \nabla_x \mathbf{f} = \nabla_x \eta + \nabla_u \eta \nabla_x \mathbf{f} \quad (1.5.196)$$

or

$$\nabla_x \mathbf{f} = -(\nabla_u \eta - \nabla_\xi g \nabla_u \xi)^{-1} (\nabla_x \eta - \nabla_\xi g \nabla_x \xi) \quad (1.5.197)$$

which is the desired formula.

We conclude this section with the important but trivial observation that symmetries are preserved under changes of coordinates, that is, if a given system of differential equations has a symmetry and then a change of coordinates will send the symmetry into a symmetry of the transformed system. This also means that the symmetry method does not depend on the coordinate system that is used to describe the given system of differential equations.

1.6 Transformation Symmetries

The notion of a symmetry can be associated with any problem and simply means a mapping of the solution of the problem into solutions of the problem. In this section the problems we will consider will be systems of ordinary or

partial differential equations *without* boundary or initial conditions and the mappings will be one parameter groups of transformations. Such symmetries are called point symmetries to distinguish them from jet symmetries that will be introduced later.

We will consider a situation where $\mathbf{u} = (u_1, \dots, u_m)$ are dependent variables and $\mathbf{x} = (x_1, \dots, x_n)$ are the independent variables. We will write the system of differential equations in the operator form

$$\mathbf{F}(\mathbf{f}) = (F_1(\mathbf{f}), \dots, F_m(\mathbf{f})) \quad (1.6.198)$$

where we are thinking of \mathbf{u} being a function \mathbf{x} ,

$$\mathbf{u} = \mathbf{f}(\mathbf{x}) . \quad (1.6.199)$$

Here each F_j is an expression in \mathbf{x} , \mathbf{f} and finite number of the derivatives

$$\frac{\partial^k f_j}{\partial x_i^k} . \quad (1.6.200)$$

The function $\mathbf{f}(\mathbf{x})$ is a solution of the system of equations provided

$$\mathbf{F}(\mathbf{f}) = 0 . \quad (1.6.201)$$

Next let $\mathbf{G}(\epsilon, \mathbf{x}, \mathbf{u})$ be a one parameter group of transformations on R^{n+m} and $\mathbf{G}(\epsilon, \mathbf{f})$ be the corresponding action surfaces $\mathbf{u} = \mathbf{f}(\mathbf{x})$. The infinitesimal of the transformation group can be written

$$S = \mathbf{s}(\mathbf{x}, \mathbf{u}) \nabla_{\mathbf{x}} + \mathbf{r}(\mathbf{x}, \mathbf{u}) \nabla_{\mathbf{u}} \quad (1.6.202)$$

and consequently the infinitesimal action on surfaces is given by

$$\mathbf{S}(\mathbf{f}) = (S_1(\mathbf{f}), \dots, S_j(\mathbf{f})) \quad (1.6.203)$$

where

$$S_j(\mathbf{f}) = -\mathbf{s}(\mathbf{x}, \mathbf{f}) \nabla_{\mathbf{x}} f_j + r_j(\mathbf{x}, \mathbf{f}) . \quad (1.6.204)$$

The requirement that $\mathbf{G}(\epsilon, \mathbf{f})$ sends solutions of $\mathbf{F}(\mathbf{f}) = 0$ into solutions of the same problem can be written

$$\mathbf{F}(\mathbf{f}) = 0 \Rightarrow \mathbf{F}(\mathbf{G}(\epsilon, \mathbf{f})) = 0 . \quad (1.6.205)$$

This is called the *group invariance condition*. The power of the infinitesimal method comes from differentiating the right hand side of the group invariance condition (1.6.205) with respects to ϵ and setting $\epsilon = 0$. Before we do this we need a definition.

Definition. The directional derivative of the functional $\mathbf{F}(\mathbf{f})$ at the point \mathbf{f} in the direction \mathbf{g} is given by

$$(D_{\mathbf{f}}\mathbf{F})(\mathbf{g}) = \frac{d}{d\epsilon}\mathbf{F}(\mathbf{f} + \epsilon\mathbf{g}) \big|_{\epsilon=0} . \quad (1.6.206)$$

This derivative is sometimes called a Frechet or Gataux derivative, especially when a notion of convergence is used in the definition.

To differentiate the expression

$$\mathbf{F}(\mathbf{G}(\epsilon, \mathbf{f})) \quad (1.6.207)$$

we use a power series expansion to write

$$\mathbf{G}(\epsilon, \mathbf{f}) \cong \mathbf{f} + \epsilon\mathbf{S}(\mathbf{f}) \quad (1.6.208)$$

and then compute

$$\frac{d}{d\epsilon}\mathbf{F}(\mathbf{G}(\epsilon, \mathbf{f})) \big|_{\epsilon=0} \cong \frac{d}{d\epsilon}\mathbf{F}(\mathbf{f} + \epsilon\mathbf{S}(\mathbf{f})) \big|_{\epsilon=0} = (D_{\mathbf{f}}\mathbf{F})(\mathbf{S}(\mathbf{f})) . \quad (1.6.209)$$

Definition. The infinitesimal group

$$L = \mathbf{s}(\mathbf{x}, \mathbf{u})\nabla_{\mathbf{x}} + \mathbf{r}(\mathbf{x}, \mathbf{u})\nabla_{\mathbf{u}} \quad (1.6.210)$$

is an infinitesimal symmetry of

$$\mathbf{F}(\mathbf{f}) = 0 \quad (1.6.211)$$

provided the corresponding infinitesimal action on surfaces $\mathbf{S}(\mathbf{f})$ satisfies

$$\mathbf{F}(\mathbf{f}) = 0 \Rightarrow (D_{\mathbf{f}}(\mathbf{F})(\mathbf{S}(\mathbf{f}))) = 0 . \quad (1.6.212)$$

This form of condition is not convenient for computations. However, the convenient form depends on the particular problem being studied so we

postpone further derivations until Chapter II and III. Before we turn to these applications, we note a few useful properties of the directional derivative.

Proposition. The directional derivative $(D_{\mathbf{f}}\mathbf{F})(\mathbf{g})$ is linear in \mathbf{g} .

Proof. Set $\alpha = \epsilon a$, $\beta = \epsilon b$ and then compute

$$\begin{aligned}
 (D_{\mathbf{f}}\mathbf{F})(a\mathbf{g} + b\mathbf{h}) &= \frac{d}{d\epsilon} F(\mathbf{f} + \epsilon(a\mathbf{g} + b\mathbf{h})) \big|_{\epsilon=0} & (1.6.213) \\
 &= \frac{d\alpha}{d\epsilon} \frac{\partial}{\partial \alpha} \mathbf{F}(\mathbf{f} + \alpha\mathbf{g} + \beta\mathbf{h}) \big|_{\epsilon=0} \\
 &\quad + \frac{d\beta}{d\epsilon} \frac{\partial}{\partial \beta} \mathbf{F}(\mathbf{f} + \alpha\mathbf{g} + \beta\mathbf{h}) \big|_{\epsilon=0} \\
 &= a(D_{\mathbf{f}}\mathbf{F})(\mathbf{g}) + b(D_{\mathbf{f}}\mathbf{F})(\mathbf{h}) .
 \end{aligned}$$

Proposition. If $\mathbf{F}(\mathbf{f})$ is linear in \mathbf{f} , then

$$(D_{\mathbf{f}}\mathbf{F})(\mathbf{g}) = \mathbf{F}(\mathbf{g}) . \quad (1.6.214)$$

Proof. Compute

$$(D_{\mathbf{f}}\mathbf{F})(\mathbf{g}) = \frac{d}{d\epsilon} \mathbf{F}(\mathbf{f} + \epsilon\mathbf{g}) \big|_{\epsilon=0} = \frac{d}{d\epsilon} \{F(\mathbf{f}) + \epsilon\mathbf{F}(\mathbf{g})\} \big|_{\epsilon=0} = \mathbf{F}(\mathbf{g}) . \quad (1.6.215)$$

Chapter 2

ORDINARY DIFFERENTIAL EQUATIONS

2.1 Introduction

We now apply the material developed in Chapter I to systems of ordinary differential equations. In this section we will describe a general system of differential equations because this is what our VAXIMA programs work with. The reader who is new to this material will find that a light reading of this section followed by a more careful reading of the next few sections will make this material more understandable. The next few sections do not explicitly depend on this section! Let $\mathbf{y} = (y_1, \dots, y_m)$ be the dependent variables and t be the independent variable. We will be interested in higher order nonlinear systems of the form

$$\frac{d^{p_i} y_i}{dt^{p_i}} = H_i(\mathbf{y}) , \quad 1 \leq i \leq m . \quad (2.1.1)$$

Here the p_i are positive integers, $p_i > 0$, and $H_i(\mathbf{y})$ is an operator function of \mathbf{y} and the derivatives of \mathbf{y} that are of lower order than the derivatives on the left hand side of the equation (2.1.1). Thus

$$H_i(\mathbf{y}) = h_i \left(t, y_1, \dots, y_m, \dots, \frac{d^{k_1} y_1}{dt^{k_1}} , \dots, \frac{d^{k_m} y_m}{dt^{k_m}} , \dots, \right) \quad (2.1.2)$$

where $k_j < p_j$ and h_j is an analytic function of its arguments. More precisely, let

$$p = \sum_{j=1}^m p_j \quad (2.1.3)$$

and then introduce the $p - m$ variables

$$v_j^k, \quad 1 \leq j \leq m, \quad 1 \leq k < p_j. \quad (2.1.4)$$

We think of v_j^k being short hand for $d^k y_j / dt^k$. Then the h_i are analytic functions of the $p + 1$ variables (t, \mathbf{y}, v_j^k) , that is,

$$h_i = h_i(t, y_1, \dots, y_m, \dots, v_j^k, \dots). \quad (2.1.5)$$

To apply the results of Section 6 of Chapter 1, let

$$\mathbf{f}(t) = (f_1(t), \dots, f_m(t)) \quad (2.1.6)$$

and then introduce the operator

$$\mathbf{F}(\mathbf{f}) = (F_1(\mathbf{f}), \dots, F_m(\mathbf{f})) \quad (2.1.7)$$

where

$$F_j(\mathbf{f}) = \frac{d^{p_j}}{dt^{p_j}} f_j - H_j(\mathbf{f}). \quad (2.1.8)$$

The derivative of \mathbf{F} in the direction \mathbf{g} is given by

$$(D_{\mathbf{f}} \mathbf{F})(\mathbf{g}) = ((D_{\mathbf{f}} F_1)(\mathbf{g}), \dots, (D_{\mathbf{f}} F_m)(\mathbf{g})) \quad (2.1.9)$$

where

$$\begin{aligned} (D_{\mathbf{f}} F_j)(\mathbf{g}) &= \frac{d}{d\epsilon} \left\{ \frac{d^{p_j}(f_j + \epsilon g_j)}{dt^{p_j}} - H_j(\mathbf{f} + \epsilon \mathbf{g}) \right\}_{\epsilon=0} \\ &= \frac{d^{p_j} g_j}{dt^{p_j}} - (D_{\mathbf{f}} H_j)(\mathbf{g}). \end{aligned} \quad (2.1.10)$$

Next,

$$\begin{aligned} (D_{\mathbf{f}} H_j)(\mathbf{g}) &= \frac{d}{d\epsilon} h_j \left(t, f_1 + \epsilon g_1, \dots, f_m + \epsilon g_m, \dots, \frac{d^k(p_j + \epsilon g_j)}{dt^k}, \dots \right) \Big|_{\epsilon=0} \\ &= \frac{\partial h_j}{\partial y_1} \cdot g_1 + \dots + \frac{\partial h_j}{\partial y_m} \cdot g_m + \dots + \frac{\partial h_j}{\partial v_j^k} \cdot \frac{d^k g_j}{dt^k} + \dots \end{aligned} \quad (2.1.11)$$

The condition of infinitesimal invariance (see Section 6 of Chapter 1) is

$$\mathbf{F}(\mathbf{f}) = 0 \Rightarrow (D_{\mathbf{f}}\mathbf{F})(\mathbf{S}(\mathbf{f})) = 0 . \quad (2.1.12)$$

Recall that S has the form

$$\begin{aligned} \mathbf{S}(\mathbf{f}) &= (S_1(f), \dots, S_m(f)), \text{ where} \\ S_j(\mathbf{f}) &= -\mathbf{S}(t, \mathbf{f}) \cdot \nabla_x f_j + r_j(t, \mathbf{f}) , \quad 1 \leq j \leq m , \end{aligned} \quad (2.1.13)$$

where $s(t, \mathbf{y})$ and $r_j(t, \mathbf{y})$, $1 \leq j \leq m$ are functions that are to be determined. The way this condition is used is to replace all derivatives of the form

$$\frac{d^{p_i+k} f_i}{dt^{p_i+k}} \quad (2.1.14)$$

that occur in

$$(D_{\mathbf{f}}\mathbf{F})(\mathbf{S}(\mathbf{f})) = 0 \quad (2.1.15)$$

by the right hand side (or appropriate derivative thereof) of one of the differential equations (2.1.1). We call the resulting expression \mathbf{E} . The expression \mathbf{E} now contains only derivatives of f_i of order less than p_i . At this point the f_i are still restricted to solutions of the system of differential equations which are not normally known. However, the existence theorem for the initial value problem says that it is always possible to find a solution of the differential equation satisfying the initial conditions

$$\frac{d^k y_j}{dt^k}(t) = v_j^k , \quad 1 \leq j \leq m , \quad 0 \leq k < p_j , \quad (2.1.16)$$

where $v_j^0 = y_j$ and the v_j^k , $1 \leq j \leq m$, $0 \leq k < p_j$ are arbitrary. Consequently the derivatives $d^k f_j / dt^k$ in the expression \mathbf{E} may be replaced by the variables v_j^k . Thus \mathbf{E} becomes an expression of the form

$$\mathbf{E}(t, y_1, \dots, y_m, v_1^1, \dots, v_m^1, \dots, v_j^k, \dots) = 0 \quad (2.1.17)$$

and the equality holds for all values of $(t, y_1, \dots, y_m, v_1^1, \dots, v_m^1, \dots, v_j^k, \dots)$. This expression also depends on the coefficients of the unknown infinitesimal symmetry operators. It is this last expression (2.1.17) that is solved for the coefficients of the infinitesimal transformation.

2.2 One First Order Equation

It is important to realize that the case of a single first order differential equation is very simple compared to other cases. Thus this section does not provide good insight into what will happen with second order equations, systems of equations or partial differential equations. However, the calculations are simple and thus this is a nice place to start, other authors have discussed this case [4, Section 1.9].

The differential equation under consideration is to be written in the form

$$y' = a(x, y) \quad (2.2.18)$$

where $y' = dy/dx$. The solutions of differential equations of this form are curves $y = f(x)$ in R^2 and consequently we will need to consider infinitesimal groups on R^2 ,

$$L = r(x, y) \frac{\partial}{\partial x} + s(x, y) \frac{\partial}{\partial y} . \quad (2.2.19)$$

The corresponding infinitesimal action on curves $y = f(x)$ is given by (see Chapter 1, Section 4)

$$S(f) = -r(x, f)f' + s(x, f) \quad (2.2.20)$$

The directional derivative of the operator form of the equation, (2.2.18)

$$F(f) = f' - a(x, f) , \quad (2.2.21)$$

in the direction g is

$$\begin{aligned} (D_f F)(g) &= \frac{d}{d\epsilon} F(f + \epsilon g) |_{\epsilon=0} \\ &= \frac{d}{d\epsilon} \{ f' + \epsilon g' - a(x, f + \epsilon g) \} |_{\epsilon=0} \\ &= g' - a_y(x, f)g . \end{aligned} \quad (2.2.22)$$

The invariance condition (see Chapter 1, Section 6)

$$F(f) = 0 \Rightarrow (D_f F)(S(f)) = 0 , \quad (2.2.23)$$

then becomes

$$F(f) = 0 \Rightarrow \frac{d}{dx} S(f) - a_y(x, f)S(f) = 0 , \quad (2.2.24)$$

that is (we now suppress function arguments),

$$f' = a(x, f) \Rightarrow -rf'' - r_x f' - r_y (f')^2 - a_y (-rf' + s) = 0 . \quad (2.2.25)$$

The next step is to use the differential equation (2.2.18) to eliminate f' and f'' from (2.2.25). The chain rule applied to $f' = a(x, f)$ gives

$$f'' = a_x + a_y f' = a_x + aa_y \quad (2.2.26)$$

and then this converts (2.2.25) to

$$-r(a_x + aa_y) - r_x a - r_y a^2 + s_x + s_y a + raa_y + sa_y = 0 . \quad (2.2.27)$$

or

$$-ra_x - r_x a - r_y a^2 + s_x + s_a + sa_y = 0 . \quad (2.2.28)$$

This equation (2.2.28) is to hold for every solution $y = f(x)$ of the given differential equation 2.2.18.

(2.2.28)

However, the existence theorem for the initial value problem for 2.2.18 says that for every x_0 and y_0 there exists a solution $y = f(x)$ of the differential equation with $y_0 = f(x_0)$. Consequently, (2.2.28) must hold for all x_0 and y_0 . If we relabel x_0 and y_0 by x and y , then (2.2.28) must hold for all x and y , that is,

$$\begin{aligned} & -r(x, y)a_x(x, y) - r_x(x, y)a(x, y) - r_y(x, y)a^2(x, y) + \\ & s_x(x, y) + s_y(x, y)a(x, y) + s(x, y)a_y(x, y) \equiv 0 . \end{aligned} \quad (2.2.29)$$

Before we start looking at examples, let us look for a moment at the converse of our standard problem. The standard problem is, of course, given a differential equation find the groups that leave the solution space of the equation invariant. The converse is, given a group find the equations whose solution space is left invariant by the group. To illustrate this we consider translation groups in either x or y .

Suppose a differential equation is invariant under the group of translations in the y variable. In this case, the infinitesimal group is given by (see Chapter 1, Section 2)

$$\frac{\partial}{\partial y} , \quad (2.2.30)$$

that is,

$$r(x, y) = 0 , \quad s(x, y) = 1 \quad (2.2.31)$$

and invariance condition (2.2.29) becomes

$$a_y(x, y) = 0 , \quad (2.2.32)$$

that is, $a = a(x)$ and then the differential equation becomes

$$y' = a(x) . \quad (2.2.33)$$

The differential equation is a simple integration problem with the solution

$$y = \int a(x) dx . \quad (2.2.34)$$

On the other hand, suppose that a differential equation 2.2.18 is invariant under the group of translations in the x variable. In this case, the infinitesimal group is given by

$$\frac{\partial}{\partial x} \quad (2.2.35)$$

or

$$r(x, y) = 1 , \quad s(x, y) = 0 \quad (2.2.36)$$

and invariance condition (2.2.29) becomes

$$a_x(x, y) = 0 . \quad (2.2.37)$$

Thus the differential equation becomes

$$y' = a(y) \quad (2.2.38)$$

which is a special case of a separable equation. An implicit solution is given by integration,

$$\int \frac{dy}{a(y)} = x . \quad (2.2.39)$$

This illustrates why we hope that transforming the differential equation to canonical coordinates for one of its invariance groups will result in a solution or simplification of the given differential equation.

Before we check this out, let us study the simplest example we can imagine. Thus we will choose $a \equiv 0$ and study the equation

$$y' = 0 . \quad (2.2.40)$$

The invariance condition (2.2.29) becomes

$$s_x = 0 . \quad (2.2.41)$$

This then implies that any infinitesimal with

$$r = r(x, y) , \ s = s(y) , \quad (2.2.42)$$

yields a group that leaves the equation (2.2.40) invariant. Written in operator notation, the infinitesimal transformations have the form

$$L = r(x, y) \frac{\partial}{\partial x} + s(y) \frac{\partial}{\partial y} , \quad (2.2.43)$$

Consequently this “simplest” differential equation has an infinite dimensional invariance group.

Although this example is very simple we have set up a program, `ode_sym_1`, to do the calculation. This program is in the file `examples.v`. We include the program listing here to illustrate the ease with which the programs can be used. For more details on the meaning of the program see the chapter on programs and the MACSYMA Manual [29]. The next two sections also include example programs and this chapter concludes with a more interesting example.

Program ode_sym_1

```

ode_sym_1() := block(
/* This program computes the symmetries of the simplest first order
   ordinary differential equation in one variable. */

/* The veryverbose mode will allow the user to see some of the inner
   working of the program. */
verbose : true,
veryverbose : true,

/* The flag num_diff is used in more complicated examples. */
num_diff:0,

/* Set the dependent and independent variables. */
dep : [y],
indep : [x],

/* Define the differential equation. Note the use of the noun form
   of the diff operator. */
diffeqn : ['diff(y,x) = 0],

/* Now load and execute the symmetries program. Note that the
   program doitall attempts a more complete solution than symmetry
   but is not appropriate for such a simple example. */
load(symmetry),
symmetry(),
end_ode_sym_1)$

```

The one parameter transformation group is obtained by exponentiating the infinitesimal,

$$\begin{pmatrix} x(\epsilon) \\ y(\epsilon) \end{pmatrix} = e^{\epsilon(r(x,y)\frac{\partial}{\partial x} + s(y)\frac{\partial}{\partial y})} \begin{pmatrix} x \\ y \end{pmatrix}. \quad (2.2.44)$$

The Differential Equation property for Lie series says that

$$\dot{x} = r(x, y), \quad \dot{y} = s(y). \quad (2.2.45)$$

Consequently, if we know s and r , then y and then x can be found by integration.

Note that the solutions of the differential equation are all horizontal lines. The transformation group can be thought of as follows. Starting at a point (x, y) first move some distance in y that is independent of x . Next, move some distance in the x direction. Thus, if two points are on some horizontal line, then both points will end up on the same horizontal line. Clearly, any such transformation sends solutions of the differential equation (2.2.40) into a solution of the differential equation.

The next example cannot be done using our programs because of certain difficulties with the differentiation routines in MACSYMA [35]. Once the differentiation routine is fixed it would be a simple matter to extend our programs to handle this type of calculation. The examples concerns finding a group of transformations that leaves the solution space of each of a *class* of differential equations invariant. This procedure is not general and works here because single first order equations have so many symmetries.

Example. The equation

$$y' = a\left(\frac{y}{x}\right) \quad (2.2.46)$$

for all functions a of a single variable is a common example in many ordinary differential equation texts. The infinitesimal invariance condition (2.2.29) for this equation is

$$s_x + s_y a - r_x a - r_y a^2 + \frac{r_y a'}{x^2} - \frac{s a'}{x} = 0 . \quad (2.2.47)$$

The calculation that follows is an excellent example of the techniques used to find infinitesimal symmetries. Because a is arbitrary we must have

$$\begin{aligned} s_x &= 0 , & r_y &= 0 , \\ s_y - r_x &= 0 , & yr - xs &= 0 . \end{aligned} \quad (2.2.48)$$

This is an *over determined* system of equations for r and s , that is, there are 4 equations and 2 unknowns. Differentiating the equation $s_y - r_x$ with respects to x and y gives

$$r_{xx} = s_{xy} = 0 , \quad s_{yy} = r_{xy} = 0 . \quad (2.2.49)$$

The condition $r_y = 0$ implies that $r = r(x)$ and then the condition $r_{xx} = 0$ implies that $r = c_1x + d_1$ for some constants c_1 and d_1 . Similarly, $s = c_2y + d_2$. Now the condition (2.2.48), $yr - xs = 0$, becomes

$$c_1xy + d_1y - c_2xy + d_2x = 0 . \quad (2.2.50)$$

This must hold for all x and y so $d_1 = 0$, $d_2 = 0$ and $c_1 = c_2$. The value of c_1 ($c_1 \neq 0$) doesn't affect the results because the symmetries form a linear space, so

$$L = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} . \quad (2.2.51)$$

This is the infinitesimal generator of the dilation or scaling group (see Chapter 1, Section 2)

$$\xi = e^\epsilon x , \quad \eta = e^\epsilon y . \quad (2.2.52)$$

This is the *only* symmetry of this *class* of equations.

We now point out that it is obvious that scalings are a symmetry of this class of equations although it is not completely obvious that this is the *only* symmetry. Set

$$\lambda = e^\epsilon \quad (2.2.53)$$

so that our notation agrees with that found many places in the literature. Then choose

$$\xi = \lambda x , \quad \eta = \lambda y \quad (2.2.54)$$

so that

$$d\xi = \lambda dx , \quad d\eta = \lambda dy . \quad (2.2.55)$$

Consequently

$$\frac{\eta}{\xi} = \frac{y}{x} , \quad \frac{d\eta}{d\xi} = \frac{dy}{dx} . \quad (2.2.56)$$

In the (ξ, η) coordinates the differential equation (2.2.46) becomes

$$\eta' = \frac{d\eta}{d\xi} = a \left(\frac{\eta}{\xi} \right) \quad (2.2.57)$$

which is the same equation as the original equation (2.2.46).

Let us now transform everything in this example to canonical coordinates (see Chapter 1, Section 5) for the dilation group. This requires the computation of functions $\xi = \xi(x, y)$, $\eta = \eta(x, y)$ such that

$$L\xi = 0 , \quad L\eta = 1 \quad (2.2.58)$$

where

$$L = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} . \quad (2.2.59)$$

The characteristic equation for L is

$$\frac{dx}{x} = \frac{dy}{y} \quad (2.2.60)$$

which has

$$\frac{y}{x} = k \quad (2.2.61)$$

as an integral. Consequently

$$\xi = h_1 \left(\frac{y}{x} \right) \quad (2.2.62)$$

for any function h_1 of a single variable. To find η we try $\eta = \eta(y)$ and then $L\eta = 1$ becomes

$$y \frac{d\eta}{dy} = 1 \quad (2.2.63)$$

or $\eta = \ln |y| + c$. Consequently

$$\eta = h_2 \left(\frac{y}{x} \right) + \ln |y| \quad (2.2.64)$$

for any function h_2 of a single variable.

We now choose (because things come out nice)

$$\begin{aligned} \eta &= \ln |y| , & y &= e^\eta , \\ \xi &= \frac{y}{x} , & x &= \frac{e^\eta}{\xi} . \end{aligned} \quad (2.2.65)$$

The differential equation (2.2.46) becomes

$$\frac{\xi^2 \eta'}{\xi \eta' - 1} = f(\xi) \quad (2.2.66)$$

or

$$\eta' = \frac{1}{\xi f(\xi) - \xi^2} . \quad (2.2.67)$$

This equation can now be solved by integration,

$$\eta = \int \frac{d\xi}{\xi f(\xi) - \xi^2} . \quad (2.2.68)$$

Exercise. Show that another choice for coordinates is

$$\eta = \frac{y}{x} + \ln |x|, \quad \xi = \frac{y}{x} \quad (2.2.69)$$

and in these coordinates the differential equation becomes

$$\eta' = 1 + \frac{1}{f(\xi) - \xi}. \quad (2.2.70)$$

By the way, the usual transformation applied to this equation is

$$\eta = \frac{y}{x}, \quad \xi = x \quad (2.2.71)$$

which then gives rise to the separable equation

$$\xi \eta' = f(\eta) - \eta. \quad (2.2.72)$$

Because our methods force the equation to become an integral we would not find this transformation.

Exercise. Show that the infinitesimal invariance condition for the class of equations

$$y' = a(x - y), \quad (2.2.73)$$

for any function a of one variable reduces to

$$\begin{aligned} s_x &= 0, & r + s &= 0, \\ r_y &= 0, & r_x - s_y &= 0. \end{aligned} \quad (2.2.74)$$

Consequently the symmetry operators are given by

$$L = \frac{\partial}{\partial x} - \frac{\partial}{\partial y}. \quad (2.2.75)$$

Canonical coordinates are given by

$$\eta = \frac{y - x}{2}, \quad \xi = \frac{y + x}{2}, \quad (2.2.76)$$

and in these coordinates the differential equation reduces to the integral

$$\frac{d\eta}{d\xi} = \frac{f(2\xi) - 1}{f(2\xi) + 1}. \quad (2.2.77)$$

The symmetry properties of a first order equation can be derived from the fact that symmetry groups are invariant under a change of variables. The equation we are studying is

$$y' = a(x, y) . \quad (2.2.78)$$

We now construct a change of variables.

$$\begin{aligned} \xi &= \xi(x, y) , & x &= x(\xi, \eta) , \\ \eta &= \eta(x, y) , & y &= y(\xi, \eta) , \end{aligned} \quad (2.2.79)$$

that will transform (2.2.78) into the equation $y' = 0$. It is important to note that we are setting things up so that y and η are dependent variables while x and ξ are independent variables. Now

$$dy = y_\xi d\xi + y_\eta d\eta , \quad dx = x_\xi d\xi + x_\eta d\eta \quad (2.2.80)$$

and consequently

$$y' = \frac{dy}{dx} = \frac{y_\xi d\xi + y_\eta d\eta}{x_\xi d\xi + x_\eta d\eta} = \frac{y_\xi + y_\eta \eta'}{x_\xi + x_\eta \eta'} . \quad (2.2.81)$$

Now solve for η' ,

$$\eta' = -\frac{x_\xi y' - y_\xi}{x_\eta y' - y_\eta} . \quad (2.2.82)$$

Thus, in the (ξ, η) coordinates the differential equation (2.2.78) becomes

$$\eta' = -\frac{x_\xi a - y_\xi}{x_\eta a - y_\eta} . \quad (2.2.83)$$

If

$$x_\xi f - y_\xi = 0 , \quad (2.2.84)$$

that is,

$$x_\xi(\xi, \eta)f(x(\xi, \eta), y(\xi, \eta)) - y_\xi(\xi, \eta) = 0 , \quad (2.2.85)$$

then the differential equation becomes

$$\eta' = 0 . \quad (2.2.86)$$

The above condition can be written

$$\frac{y_\xi}{x_\xi} = f(x(\xi, \eta), y(\xi, \eta)) . \quad (2.2.87)$$

This equation says that if η is fixed and ξ varies, then $(x(\xi, \eta), y(\xi, \eta))$ is a parametric curve which is a solution of the differential equation $y' = f(x, y)$. Another way to describe this curve is that it is a level curve of η ,

$$\eta(x, y) = \text{const.} \quad (2.2.88)$$

Thus $\eta(x, y)$ must be an integral of the equation (2.2.78). On the other hand, if $F(x, y)$ is an integral of (2.2.78), then we can choose $\eta = F(x, y)$ and $\xi = \xi(x, y)$ as new coordinates, where $\xi(x, y)$ is fairly arbitrary, and in this coordinate system the differential equation (2.2.78) becomes

$$\eta' = 0. \quad (2.2.89)$$

Previously we found the infinitesimal symmetries and one parameter groups of symmetries of $y' = 0$. However, if we want the equation $y' = 0$ to transform into $\eta' = 0$, then equation (2.2.83) tells us that we must have $y_\xi = 0$. Thus any transformation of the form

$$x = x(\xi, \eta), \quad y = y(\eta), \quad (2.2.90)$$

is a symmetry of $y' = 0$. Interchanging the roles of (x, y) and (ξ, η) and then solving for x and y gives

$$\eta = \eta(y), \quad \xi = \xi(x, y) \quad (2.2.91)$$

which is the analog of the previously derived condition (2.2.45) on the symmetries. Because symmetry groups are preserved under changes of variables, the equation $y' = a(x, y)$ must have infinitely many symmetries.

2.3 One Second Order Equation

The case of a second order equation is fairly typical of the general case of higher order equations and partial differential equations. In particular, the result that the infinitesimal symmetries form a finite dimensional space is important and typical. We will consider an equation of the form

$$y'' = a(x, y, y'). \quad (2.3.92)$$

Of course, $y' = dy/dx$ and $y'' = dy'/dx$ and a is an arbitrary function of three variables $a = a(x, y, v)$.

The operator form of the differential equation is

$$F(f) = f'' - a(x, f, f') \quad (2.3.93)$$

and the derivative of F in the direction g is

$$\begin{aligned} (D_f F)(g) &= \frac{d}{d\epsilon} F(f + \epsilon g) \big|_{\epsilon=0} \\ &= \frac{d}{d\epsilon} (f + \epsilon g)'' - a(x, f + \epsilon g, (f + \epsilon g)') \big|_{\epsilon=0} \\ &= g'' - a_y(x, f, f')g - a_v(x, f, f')g' \end{aligned} \quad (2.3.94)$$

The infinitesimal symmetries have the same form as in the first order equation case,

$$L = r(x, y) \frac{\partial}{\partial x} + s(x, y) \frac{\partial}{\partial y} . \quad (2.3.95)$$

The corresponding infinitesimal action on curves is given by

$$S(f) = -r(x, f)f' + s(x, f) . \quad (2.3.96)$$

Now, the invariance condition (see Chapter 1, Section 6) becomes

$$F(f) = 0 \Rightarrow (D_f F)(S(f)) = 0 , \quad (2.3.97)$$

that is,

$$\begin{aligned} f'' - a(x, f, f'') &= 0 \Rightarrow \\ (-r(x, f)f' + s(x, f))'' - a_y(x, f, f')(-r(x, f)f' + s(x, f)) \\ - a_v(x, f, f')(-r(x, f)f' + s(x, f))' &= 0 . \end{aligned} \quad (2.3.98)$$

Now

$$(-rf' + s)' = -r_x f' - r_y (f')^2 - r f'' + s_x + s_y f' \quad (2.3.99)$$

and

$$\begin{aligned} (-rf' + s)'' &= -r_{xx} f' - 2r_{xy} (f')^2 - 2r_x f'' - r_{yy} (f')^3 - 3r_y f' f'' \\ &\quad - r f''' + s_{xx} + 2s_{xy} f' + s_{yy} (f')^2 . \end{aligned} \quad (2.3.100)$$

The differential equation (2.3.92) gives $f'' = a(x, f, f')$ and differentiating this gives

$$f''' = a_x(x, f, f') + a_y(x, f, f')f' + a_v(x, f, f')f'' . \quad (2.3.101)$$

Combining (2.3.98) through (2.3.101)) gives

$$\begin{aligned} & a_v(-s_x - s_y y' + r_x y' + r_y (y')^2) + \\ & a_y(-s - r y') + a_x(-r) + a(s_y - 2r_x - 3r_y f') + \\ & s_{xx} - r_{xx} y' + 2s_{xy} y' - 2r_{xy} (y')^2 + s_{yy} (y')^2 - r_{yy} (y')^3 = 0 . \end{aligned} \quad (2.3.102)$$

This equation holds for all x and all solutions $f(x)$ of the differential equation (2.3.92).

The fact that the initial value problem

$$f'' = a(x, f, f') , \quad f(x) = y , \quad f'(x) = v , \quad (2.3.103)$$

always has a solution allows us to replace f by y and f' by v in (2.3.102),

$$\begin{aligned} & a_v(x, y, v)(-s_x(x, y) - s_y(x, y)v + r_x(x, y)v^2) \\ & + a_y(x, y, v)(-s(x, y) - r(x, y)v) + a_x(x, y)(-r(x, y)) \\ & + a(x, y, v)(s_y(x, y) - 2r_x(x, y) - 3r_y(x, y)v) + s_{xx}(x, y) \\ & - r_{xx}(x, y)v + 2s_{xy}(x, y)v - 2r_{xy}(x, y)v^2 + s_{yy}(x, y)v^2 - r_{yy}(x, y)v^3 = 0 \end{aligned} \quad (2.3.104)$$

Equation (2.3.104) must hold for all (x, y, v) . It is also helpful to collect the terms in the following way,

$$\begin{aligned} & s_{xx} + 2vs_{xy} + v^2s_{yy} - vr_{xx} - 2v^2r_{xy} - v^3r_{yy} \\ & + r_x(va_v - 2a) + r_y(v^2a_v - 3va) + r(-va_y - a_x) \\ & + s_x(-a_v) + s_y(-va_v + a) + s(-a_y) = 0 . \end{aligned} \quad (2.3.105)$$

Thus we see that) (2.3.105) is linear in a . Consequently, the inverse problem of finding an equation that is invariant under a given group becomes a problem of solving a first order linear partial differential equation for $a = a(x, y, v)$. The existence theory for this type of equation tells us that there are infinitely many solutions. However, when the differential equation is given, that is, $a = a(x, y, v)$ is given, then (2.3.105) is a single second order linear homogeneous partial differential equation for determining $r(x, y)$ and $s(x, y)$. Because $r(x, y)$ and $s(x, y)$ do not depend on v , this equation implies much more. If $a(x, y, v)$ is expanded as a power series in v and this is substituted into (2.3.105), then the coefficients of the powers of v must be zero. This will produce an infinite set of linear homogeneous second order (at most) partial differential equations for determining $r(x, y)$ and $s(x, y)$. Because the explicit

powers of v in (2.3.105) range between 0 and 3 and the second order terms do depend on a , this system must contain at least 4 independent equations for r and s . Consequently the system is overdetermined implying that there is at most a finite number of symmetries [56, 18].

Example. Let us look at the simplest possible example,

$$y'' = 0 . \quad (2.3.106)$$

Thus $a \equiv 0$ and equation (2.3.105) becomes

$$-vr_{xx} - 2v^2r_{xy} - v^3r_{yy} + s_{xx} + 2vs_{xy} + v^2s_{yy} = 0 . \quad (2.3.107)$$

This is a polynomial in v so its coefficients must be zero,

$$\begin{aligned} r_{yy} &= 0 , & -2r_{xy} + s_{yy} &= 0 \\ -r_{xx} + 2s_{xy} &= 0 , & s_{xx} &= 0 . \end{aligned} \quad (2.3.108)$$

This last system of equations (2.3.108) is an *over determined* system of equations for r and s , that is, there are 2 unknowns and 4 equations. The following discussion illustrates an important technique for solving such equations.

Differentiate the second equation in (2.3.108) with respects to y ,

$$s_{yyy} = 2r_{xyy} = 0 , \quad (2.3.109)$$

and the third equation in with respects to x ,

$$r_{xxx} = 2s_{xxy} = 0 . \quad (2.3.110)$$

The technique for solving these equations is discussed in Appendix A. The equations

$$r_{yy} = 0 , \quad r_{xxx} = 0 \quad (2.3.111)$$

imply that

$$r = c_1x^2y + c_2xy + c_3y + c_4x^2 + c_5x + c_6 \quad (2.3.112)$$

where c_i are constants. The conditions

$$s_{xx} = 0 , \quad s_{yyy} = 0 \quad (2.3.113)$$

imply that

$$s = d_1xy^2 + d_2xy + d_3x + d_4y^2 + d_5y + d_6 , \quad (2.3.114)$$

where the d_i are constants. Plugging (2.3.114) and (2.3.112) into the middle two equations in (2.3.108) gives

$$-2(2c_1x + c_2) + (2d_1x + 2d_4) = 0 , \quad -(2c_1y + c_4) + 2(2d_1y + d_2) = 0 . \quad (2.3.115)$$

This must hold for all x and y and consequently

$$c_1 = c_4 = d_1 = d_4 = 0 , \quad (2.3.116)$$

and then

$$L = (c_2xy + c_3y + c_5x + c_6)\frac{\partial}{\partial x} + (d_2xy + d_3x + d_5y + d_6)\frac{\partial}{\partial y} . \quad (2.3.117)$$

Thus the infinitesimal symmetries of the equation $y'' = 0$ form an 8 dimensional linear space called the projective algebra. In the case of partial differential equations we will find similar algebras. The operators in this algebra are easy to exponentiate using Lie series, see [4, Section 1.7] for the details of the exponentiation using classical methods.

We have included in our example programs a program, `ode_sym_2`, to do the above computation. This program is contained in the file `examples.v`. The program uses a slightly different notation where the symmetry operator has the form

$$a_2y' + a_1 \quad (2.3.118)$$

and consequently

$$a_2 = -r , \quad a_1 = s . \quad (2.3.119)$$

This is an example that anyone wishing to use the programs should run. Note that in this elementary example the program will produce the symmetries with no help from the user. The program uses a slightly different notation for the various functions, however, comparing the program output with our discussion will make all conventions transparent. For a more complete discussion of the programming details see the chapter of programs and the last section of this chapter.

Program ode_sym_2

```
ode_sym_2() := block(  
  /* This program computes the symmetries of the simplest second  
    order ordinary differential equation in one variable. */  
  
  /* The veryverbose mode will allow the user to see some of the inner  
    working of the program. */  
  verbose : true,  
  veryverbose : true,  
  
  /* The flag num_diff is used to limit the number of differentiations  
    made in attempting a solution of the equation list. */  
  num_diff:3,  
  
  /* Set the dependent and independent variables. */  
  dep : [y],  
  indep : [x],  
  
  /* Define the differential equation. Note the use of the noun form  
    of the diff operator. */  
  diffeqn : ['diff(y,x,2) = 0],  
  
  /* Now load and execute the program doitall. The program doitall attempts  
    a more complete solution than the program symmetry. */  
  load(doitall),  
  doitall(),  
  end_ode_sym_2)$
```

2.4 Two First Order Equations

We will write the system of equations in the form

$$x' = a(t, x, y), \quad y' = b(t, x, y) \quad (2.4.120)$$

where $x' = dx/dt$ and $y' = dy/dt$. The operator form of these equations is given by

$$\mathbf{F}(f, g) = (f' - a(t, f, g), \quad g' - b(t, f, g)) \quad (2.4.121)$$

where $f = f(t)$, $g = g(t)$. The derivative of \mathbf{F} in the direction (u, v) is

$$\begin{aligned} (D_{f,g}\mathbf{F})(u, v) &= \frac{d}{d\epsilon}\mathbf{F}(f + \epsilon u, g + \epsilon v) \big|_{\epsilon=0} \\ &= (u' - a_x(t, f, g)u - a_y(t, f, g)v, \\ &\quad v' - (a_x(t, f, g)u - b_y(t, f, g)v) . \end{aligned} \quad (2.4.122)$$

The infinitesimal symmetries have the form

$$L = q(t, x, y)\frac{\partial}{\partial t} + r(t, x, y)\frac{\partial}{\partial x} + s(t, x, y)\frac{\partial}{\partial y} \quad (2.4.123)$$

and the corresponding action on curves $(f(t), g(t))$ is given by

$$\mathbf{S}(f, g) = (-qf' + r, -qg' + s) . \quad (2.4.124)$$

The invariance condition (see Chapter 1, Section 6) becomes

$$\begin{aligned} \mathbf{F}(f, g) = 0 &\Rightarrow \\ (D_{f,g}\mathbf{F})(\mathbf{L}(f, g)) &= \\ ((-qf' + r)' - (-qf' + r)a_x - (-qg' + s)a_y + \\ (-qg' + s)' - (qf' + r)b_x - (qg' + s)b_y) &= 0 , \end{aligned} \quad (2.4.125)$$

that is,

$$\begin{aligned} (-qf' + r)' - (-qf' + r)a_x - (-qg' + s)a_y &= 0 , \\ (-qg' + s)' - (qf' + r)b_x - (qg' + s)b_y &= 0 . \end{aligned} \quad (2.4.126)$$

The system of differential equations gives

$$f' = a(t, f, g) , \quad g' = b(t, f, g) \quad (2.4.127)$$

and differentiating with respects to t gives

$$\begin{aligned} f'' &= a_t + a_x f' + a_y g' = a_t + a a_x + b a_y , \\ g'' &= b_t + b_x f' + b_y g' = b_t + a b_x + b b_y . \end{aligned} \quad (2.4.128)$$

Combining (2.4.126) through (2.4.128) gives

$$\begin{aligned} a q_t + a^2 q_x + a b q_y + a_t q + a_x r + a_y s - r_t - a r_x - b r_y &= 0 , \\ b q_t + a b q_x + b^2 q_y + b_t q + b_x r + b_y s - s_t - a s_x - b s_y &= 0 . \end{aligned} \quad (2.4.129)$$

Here a , b , q , r , and s are all functions of (t, x, y) . Consequently, (2.4.129) is a system of 2 first order linear partial differential equations for determining q , r , and t . Thus the system (2.4.129) is under determined and will have infinitely many solutions.

Example. Again, we look at the simplest possible example,

$$x' = 0, \quad y' = 0. \quad (2.4.130)$$

The invariance condition (2.4.129) becomes

$$r_t = 0, \quad s_t = 0. \quad (2.4.131)$$

The solutions of (2.4.130) are lines in (t, x, y) space that are parallel to the t axis. The exponential of an infinitesimal satisfying (2.4.131) will move any point (t, x, y) to some point $(\tilde{t}, \tilde{x}, \tilde{y})$ where \tilde{x} and \tilde{y} are independent of t and then move this point parallel to the t axis, see the Differential Equation Property in Section 3 of Chapter 1 and Section 2 of this chapter. Such a transformation clearly sends solutions into solutions. The following program will do this example.

Again, we wrote a VAXIMA program `ode_sym_11`, contained in the file `examples.v`, that will do this example. As before, a comparison of the discussion in this section with the output of the program will make the notation clear. For more details see the chapter on programming and the next section of this chapter.

Program ode_sym_11

```

ode_sym_11() := block(
/* This program finds the symmetries of the simplest system of
   two first order ordinary differential equations. */

/* The veryverbose mode will allow the user to see some of the inner
   working of the program. */
verbose : true,
veryverbose : true,

/* The flag num_diff is used in more complicated examples. */
num_dif:0,

/* Set the dependent and independent variables. */
dep : [x,y],
indep : [t],

/* Define the differential equation. Note the use of the noun form
   of the diff operator. */
diffeqn : ['diff(x,t) = 0 , 'diff(y,t) = 0],

/* Now load and execute the program symmetry. Note that the program
   doitall attempts a more complete solution than symmetry but is not
   appropriate for such a simple example. */
load(symmetry),
symmetry(),
end_ode_sym_11)$

```

Example. The equation $y'' = 0$ can be converted to the equivalent system

$$x' = 0, \quad y' = x. \quad (2.4.132)$$

We will now apply the results of this section to this system and then compare this to the results on second order equations. Thus $a(t, x, y) = 0$ and $b(t, x, y) = x$ and the determining equations (2.4.129) become

$$\begin{aligned} r_t + x r_y &= 0 \\ x q_t + x^2 q_y + r - s_t - x s_y &= 0 \end{aligned} \quad (2.4.133)$$

The method of characteristics will allow us to construct infinitely many solutions to (2.4.133) . To compare this result to that obtained for second order equations, eliminate r from (2.4.133) because r determines the change in x which corresponds to the derivative of y and this was not included in the second order case. Differentiate the last equation in (2.4.133) with respects to t and y ,

$$\begin{aligned} xq_{tt} + x^2q_{ty} + r_t - s_{tt} - xs_{ty} &= 0 , \\ xq_{ty} + x^2q_{yy} + r_y - s_{ty} - xs_{yy} &= 0 . \end{aligned} \quad (2.4.134)$$

Now solve (2.4.134) for r_t and r_y and plug into the first of (2.4.133)

$$-s_{tt} + (q_{tt} + 2s_{ty})x + (2q_{ty} - s_{yy})x^2 + q_{yy}x^3 = 0 . \quad (2.4.135)$$

If we assume that s and q are independent of x , then (2.4.135) yields 4 equations that are the same as those obtained for $y'' = 0$ in the previous section. The equation for r in (2.4.133) is the standard extension of a transformation to derivatives, see Chapter 4.

Exercise. An autonomous system of equations

$$x' = a(x, y) , \quad y' = b(x, y) \quad (2.4.136)$$

can be reduced to the single first order equation

$$\frac{dy}{dx} = \frac{b(x, y)}{a(x, y)} . \quad (2.4.137)$$

Apply the theory in this section to the system (2.4.136) and the results on first order equations to (2.4.137) and then make a comparison!

2.5 The Toda Lattice

Here we wish to provide the user with a nontrivial example that will illustrate the power of our symbol manipulation programs. We decided on the Toda lattice equations because of the current interest in this system. This problem is sufficiently complex so as to be unpleasant to do by hand but not so complex that it produces a hard to understand example. It is known that the Toda lattice has a nontrivial symmetry. However, this symmetry depends on the derivative of the solutions of the differential equations (momenta) and consequently is not a point symmetry. Thus our programs will not find this symmetry. On the other hand, the Toda lattice is an autonomous system and consequently time translations will be a symmetry of the system. Thus our programs must produce this symmetry. The theory of Hamiltonian systems implies that there no other Hamiltonian symmetries (symmetries that are canonical transformations). We show, in fact, that translations are the only point symmetries of the Toda lattice.

The remainder of this section consists of a severely edited VAXIMA output. To remain in the domain of point symmetries we use the two second order equations that model the Toda lattice. To give the reader some idea of the size of this problem, it is worth noting that the file that contained all of the output from the VAXIMA session where we calculated the symmetries contained 5096 lines and the problem required 95 cpu minutes to run on a VAX11/780. The program can be found in the file examples.v. We will return to this example in a later section.

This section will not be very understandable with out some prior knowledge of MACSYMA [29]. The chapter on programs can be used to look up a description of our programs. As in our other examples we build a file that contains the information necessary to do the computation and then load this file into VAXIMA.

VAXIMA Output

```
We begin by loading a file.
```

```
(c2) load(toda);
```

```
Batching the file toda.v
```

```
(c3) ode_toda() := block(
```

```
/* The Toda lattice. */
```

```

/* The veryverbose mode will allow the user to see some of the
   inner working of the program. */
    verbose:true,
    veryverbose : true,

/* The flag num_diff is used to control the equation solver. */
    num_dif:2,

/* Set the dependent and independent variables. */
    dep:[x,y],
    indep:[t],

/* The Toda lattice is a Hamiltonion system so we use
   that notation. */
    H:(exp(2*y+2*sqrt(3)*x)+exp(2*y-2*sqrt(3)*x)
        +exp(-4*y))/24 - 1/8,

/* Define the differential equation. */
    diffeqn:[
        'diff(x,t,2) = -diff(H,x),
        'diff(y,t,2) = -diff(H,y)   ],

/* Now load and execute the program doitall. */
    load(doitall),
    doitall(),
    end_toda)$

```

Batching done.

```
(c5) ode_toda();
```

Putting the differential equations in standard form.

$$\begin{array}{c}
 \frac{d^2 x}{dt^2} = \frac{2 y^2 - 2 \sqrt{3} x}{12} - \frac{2 y + 2 \sqrt{3} x}{12} \sqrt{3} e^{\sqrt{3} y} \\
 \text{(e8)} \quad \frac{d^2 y}{dt^2} = \frac{2 y^2 - 2 \sqrt{3} x}{12} - \frac{2 y + 2 \sqrt{3} x}{12} \sqrt{3} e^{\sqrt{3} y}
 \end{array}$$

$$(e9) \quad \frac{d^2 y}{dt^2} = - \frac{2 y + 2 \sqrt{3} x}{12} e^{\frac{2 y - 2 \sqrt{3} x}{12}} + \frac{-4 y}{6} e^{\frac{2 y - 2 \sqrt{3} x}{12}}$$

Creating the symmetry operators.
The coefficients a1, a2 and
a3 are unknown functions of the variables (x,y,t).

$$1 \quad \frac{dx}{dt} = a3 \frac{dx}{dt} + a1$$

$$2 \quad \frac{dy}{dt} = a3 \frac{dy}{dt} + a2$$

When the veryverbose flag is true the program
prints a running commentary. Here are some
of the comments.

Computing the determining equations.

Eliminating the time derivatives.

Collecting the determining equations.

Cleaning up a list.

The length of the list is 2 .

Calculating the coefficients of a polynomial.

Preparing the equation list for printing.

The number of equations is 18 .

Ordering a list of length 18 .

$$(e13) \quad \frac{d^2 a1}{dy^2} = 0$$

$$(e14) \quad \frac{\frac{d^2 a3}{dy^2}}{2} = 0$$

$$(e15) \quad \frac{\frac{d^2 a3}{dx dy}}{2} = 0$$

$$(e16) \quad \frac{\frac{d^2 a3}{dx^2}}{2} = 0$$

$$(e17) \quad \frac{\frac{d^2 a2}{dx^2}}{2} = 0$$

$$(e18) \quad 2 \frac{\frac{d^2 a3}{dt dy}}{2} + 2 \frac{\frac{d^2 a1}{dx dy}}{2} = 0$$

$$(e19) \quad 2 \frac{\frac{d^2 a3}{dt dx}}{2} + \frac{\frac{d^2 a1}{dx^2}}{2} = 0$$

$$(e20) \quad 2 \frac{\frac{d^2 a3}{dt dy}}{2} + \frac{\frac{d^2 a2}{dx^2}}{2} = 0$$

$$\begin{aligned}
 & \frac{dy}{dt} \\
 (e21) \quad & 2 \frac{d^2 a3}{dt dx} + 2 \frac{d^2 a2}{dx dy} = 0 \\
 (e22) \quad & - \frac{\frac{da3}{dy} 2y + 2\sqrt{3} x}{6} + \frac{\frac{da3}{dy} 2y - 2\sqrt{3} x}{6} \\
 & + 2 \frac{d^2 a1}{dt dy} = 0
 \end{aligned}$$

The remaining equations in the equation list become progressively more complicated. Here are the functional dependencies.

$$[a1(x, y, t), a2(x, y, t), a3(x, y, t), x(t), y(t)]$$

What follows are comments printed (veryverbose:true)
as the program does its work.
Starting the solution of the equation list.
Collecting all one term equations.
Length of list is 15 .
Ordering a list of length 5 .
Solving all one term equations.
Fixing up a one term equation.

$$\text{Attempting to solve } \frac{d^2 a1}{dy^2} = 0 .$$

Ordering a list of length 10 .

$$(e33) \quad \frac{2}{dx} \frac{da5}{dx} y + \frac{2}{dx} \frac{da4}{dx} + 2 \frac{da10}{dt} = 0$$

$$dx \qquad dx \qquad dx$$

$$(e40) \quad \frac{1}{1} = a8 \frac{--}{dt} y + a5 y + a10 x \frac{--}{dt} + a9 \frac{--}{dt} + a4$$

$$(e41) \quad \frac{1}{2} = a8 y \frac{dy}{dt} + a10 x \frac{dy}{dt} + a9 \frac{dy}{dt} + a12 x + a11$$

[a1(x, y, t), a2(x, y, t), a3(x, y, t), a4(x, t), a5(x, t),
a6(x, t), a7(x, t), a8(t), a9(t), a10(t), a11(t, y), a12(t, y),
x(t), y(t)]

The next strategy for solving the equation list is to
differentiate some of the simpler equations in the list and
look for more one term equations.

Differentiating the equation list for the first time.

Differentiating a list of length 10 .

Collecting all one term equations.

Length of list is 16 .

Ordering a list of length 2 .

Solving all one term equations.

Fixing up a one term equation.

$$\text{Attempting to solve } \frac{d^2 a5}{dx^2} = 0 .$$

The program goes on this way until it has differentiated the
equation list 3 times and solved all one term equations. We note
that the third differentiation does not produce any new solvable
equations.

Preparing the equation list for printing.

The number of equations is 26 .

Cleaning up a list.

The length of the list is 26 .

Ordering a list of length 26 .

$$(e74) \quad 2 \frac{da8}{dt} + 2 a14 = 0$$

$$(e75) \quad 2 a16 + 2 \frac{da10}{dt} = 0$$

As before the equation list goes on with the equations becoming more and more complicated. The symmetry operators are now taking on a distinctly interesting form.

$$(e94) \quad \frac{1}{2} = a8 \frac{dx}{dt} y + a14 x y + a13 y + a10 x \frac{dx}{dt} + a9 \frac{dx}{dt}^2 + a19 x^2 + a18 x + a17$$

$$(e95) \quad \frac{1}{2} = a8 y \frac{dy}{dt} + a10 x \frac{dy}{dt} + a9 \frac{dy}{dt}^2 + a22 y^2 + a16 x y + a21 y + a15 x + a20$$

Here are the functional dependencies.

[a1(x, y, t), a2(x, y, t), a3(x, y, t), a4(x, t), a5(x, t),
a6(x, t), a7(x, t), a8(t), a9(t), a10(t), a11(t, y), a12(t, y),
a13(t), a14(t), a15(t), a16(t), a17(t), a18(t), a19(t),
a20(t), a21(t), a22(t), x(t), y(t)]

The symmetry program now gives up!

```
(d95)                                     end_toda
```

Looking back at the equation list we see that the first four equations can be solved for one of the unknown functions. We now do this interactively.

```
(c99) globalsolve:true$
```

```
(c102) linsolve(eqnlist[1],a14);
```

```
(d102)                                     da8
[a14 = - ---]
                                     dt
```

We proceed to solve the first 4 equations in the equation list. Next plug the solutions back into the equation list to see if there are any more simple equations.

```
(c113) eqnlist:cleanup(eqnlist);
```

Cleaning up a list.
The length of the list is 20 .
Now lets see what we have. Note that the equation list consists of expression that are to be set equal to zero.

```
(c114) first(eqnlist);
```

```
(d114) - 
$$\frac{\sqrt{3}a8^2 y + 2 \sqrt{3} x}{6} + \frac{\sqrt{3}a8^2 y - 2 \sqrt{3} x}{6}$$

```

$$- 2 \frac{d^2 a8}{dt^2} x + 2 \frac{da13}{dt}$$

The functions a8 and a13 depend only on the variable t. Consequently

differentiating the previous equation with respects to x or y will kill some of the terms in the equation.

```
(c116) diff(d114,x);
```

$$(d116) \quad -a^8 \frac{2y + 2\sqrt{3}x}{e} - a^8 \frac{2y - 2\sqrt{3}x}{e} - 2 \frac{d^2 a^8}{dt^2}$$

```
(c117) diff(%,x);
```

$$(d117) \quad 2\sqrt{3}a^8 \frac{2y - 2\sqrt{3}x}{e} - 2\sqrt{3}a^8 \frac{2y + 2\sqrt{3}x}{e}$$

The last equation impliest that:

```
(c119) a8:0$
```

We apply the same trick to the second equation in the equation list. This reduces the first two equation in the equation list to one term equations so we take care of them.

```
(c129) alloneterm(eqnlist);
Collecting all one term equations.
Length of list is 6 .
Ordering a list of length 2 .
Solving all one term equations.
Fixing up a one term equation.
```

$$\text{Attempting to solve } \frac{da13}{dt} = 0 .$$

The second one term equation is done in the same way. We now start to manipulate the third equation in the equation list in hopes of finding a simple equation. At first, the output is messy so we do not display it.

```
(c136) expand(eqnlst[3]*exp(-2*y));
(c137) diff(%,y);
(c138) diff(%,x);
(c139) diff(%,y);
```

$$(d139) \quad \frac{d^2 a18}{dt^2} - 2 y^2 e^{\frac{1}{4} y^2}$$

```
(c141) %*exp(2*y);
(c143) alloneterm(%);
Collecting all one term equations.
Length of list is 1 .
Ordering a list of length 1 .
Solving all one term equations.
Fixing up a one term equation.
```

$$\text{Attempting to solve } \frac{d^2 a18}{dt^2} = 0 .$$

This should have simplified the first equation in the equation list.

```
(c146) first(eqnlst);
```

$$(d146) \quad \frac{d^2 a9}{dt^2} + 2 k4$$

Some of our programs depend on recognizing the constants that are introduced so we use the function newc(k) to add the constant to the integral of the previous equation. Note that a9 depends only

on t so that it is not necessary to add an arbitrary function of x and y.

```
(c148) integrate(%,t)+newc(k);
```

$$(d148) \quad 2 k_4 t + \frac{da_9}{dt} + k_5$$

```
(c149) integrate(%,t)+newc(k);
```

$$(d149) \quad k_4 t^2 + k_5 t + a_9 + k_6$$

```
(c150) linsolve(%,a9);
```

$$(d150) \quad [a_9 = -k_4 t^2 - k_5 t - k_6]$$

We proceed in a fashion similar to the last few computations until we have solved for all of the "a" functions. Now all that is left to do is to solve for the "k" constants. The equation list consists of two complicated equations so we compute the first few terms of the power series expansion of the equations.

```
(c189) subst(0,x,eqnlist);
```

```
(c190) subst(0,y,%);
```

$$(d190) \quad [k_{10} t^2 + k_9 t + k_8 + 2 k_{10}, k_{12} t + k_{11}]$$

```
(c191) k10:0$k9:0$k8:0$k12:0$k11:0$
```

```
(c201) diff(subst(0,x,eqnlist),y);
```

$$(d201) \quad [k_1 - k_1 e^{-6y},$$

$$\begin{aligned}
& - 4 k_4 t y e^{-6y} + 2 k_7 y e^{-6y} + \frac{11 k_4 t e^{-6y}}{3} \\
& + \frac{k_7 e^{-6y}}{6} + 2 k_5 e^{-6y} + \frac{k_4 t}{3} - \frac{k_7}{6}]
\end{aligned}$$

(c202) k1:0\$

We now just list the input lines that are used to finish the computation.

```

(c203) subst(0,y,last(d201));
(c204) k4:0$
(c205) k5:0$
(c206) eqnlist:ev(eqnlist);
(c207) subst(0,x,diff(eqnlist,x));
(c208) k2:0$
(c209) k7:0$
(c210) eqnlist:ev(eqnlist);
(c211) k3:0$

```

The equation list is solved!

Now print out the results of the computation.

```
(c213) results();
```

Here are the only symmetries!

$$\begin{aligned}
\text{(e213)} \quad & \frac{1}{1} = -k_6 \frac{dx}{dt}
\end{aligned}$$

$$\begin{aligned}
\text{(e214)} \quad & \frac{1}{2} = -k_6 \frac{dy}{dt}
\end{aligned}$$

(d214)

done

When the symmetry operators are written as a transformation group on R^3 labeled with the variables (x, y, t) , they become constant multiples of the single infinitesimal transformation

$$\frac{\partial}{\partial t}.$$

As was mentioned at the beginning of this section, time translations are the only point symmetries of the Toda lattice. Thus we see that it is important to generalize the notion of symmetry.

Chapter 3

PARTIAL DIFFERENTIAL EQUATIONS

3.1 Introduction

The material in this section is the basis for our programs that compute point symmetries. As in Chapter 2, it is perhaps best to give this section a light reading and then turn to the next section where these ideas are applied to the heat equation. It is not necessary to understand this section to understand the next section. The third section uses our computer code to find the symmetries of Burger's equation. The method for partial differential equations is similar to the method for ordinary differential equations. Let $\mathbf{u} = (u_1, \dots, u_m)$ be the dependent variables while t and $\mathbf{x} = (x_1, \dots, x_n)$ are the independent variables. The problems of interest are nonlinear partial differential equations that can be written in the form

$$\frac{d^{p_i} u_i}{dt^{p_i}} = H_i(\mathbf{u}) , \quad 1 \leq i \leq m , \quad (3.1.1)$$

where p_i are positive integers, $p_i > 0$, and $H_i(\mathbf{u})$ is operator notation for a function of \mathbf{u} and derivatives of \mathbf{u} that are of lower order than the derivatives on the left hand side of the equation (3.1.1). Thus

$$H_i(\mathbf{u}) = h_i(t, \mathbf{x}, \mathbf{u}, \frac{\partial u_1}{\partial t}, \frac{\partial u_1}{\partial x_1}, \dots) \quad (3.1.2)$$

where h_i is a function of a finite number of variables. More precisely, if we introduce the new variables

$$u_i^{(j,\mathbf{k})}, \mathbf{k} = (k_1, \dots, k_n), 1 \leq j < p_j, 1 \leq i \leq m, \quad (3.1.3)$$

then h_i should be an analytic function of its arguments,

$$h_i = h_i(t, x, \mathbf{u}, \dots, u_i^{(j,\mathbf{k})}, \dots). \quad (3.1.4)$$

There are no restrictions on the \mathbf{k} indicies. Here we think of the superscripted u variables as short hand for derivatives,

$$u_i^{(j,\mathbf{k})} = \frac{\partial^j}{\partial t^j} \frac{\partial^{\mathbf{k}}}{\partial \mathbf{x}^{\mathbf{k}}} u_i \quad (3.1.5)$$

where

$$\frac{\partial^{\mathbf{k}}}{\partial \mathbf{x}^{\mathbf{k}}} = \frac{\partial^{k_1}}{\partial x_1^{k_1}} \cdots \frac{\partial^{k_n}}{\partial x_n^{k_n}}. \quad (3.1.6)$$

To apply the results of Section 6 of Chapter I, let

$$\mathbf{f}(t, \mathbf{x}) = (f_1(t, \mathbf{x}), \dots, f_m(t, \mathbf{x})) \quad (3.1.7)$$

(here we think of $\mathbf{u} = \mathbf{f}(t, \mathbf{x})$ as solutions) and then introduce the operator

$$\mathbf{F}(\mathbf{f}) = (F_1(\mathbf{f}), \dots, F_m(\mathbf{f})) \quad (3.1.8)$$

where

$$F_j(\mathbf{f}) = \frac{\partial^{p_j}}{\partial t^{p_j}} f_j - H_i(\mathbf{f}). \quad (3.1.9)$$

The derivative of \mathbf{F} in the direction \mathbf{g} is given by

$$(D_{\mathbf{f}}\mathbf{F})(\mathbf{g}) = ((D_{\mathbf{f}}F_1)(\mathbf{g}), \dots, (D_{\mathbf{f}}F_m)(\mathbf{g})) \quad (3.1.10)$$

where

$$\begin{aligned} (D_{\mathbf{f}}F_j)(\mathbf{g}) &= \frac{d}{d\epsilon} \left\{ \frac{d^{p_j}}{dt^{p_j}} (f_j + \epsilon g) - H_i(f + \epsilon g) \right\}_{\epsilon=0} \\ &= \frac{d^{p_j}}{dt^{p_j}} g_j - (D_{\mathbf{f}}H_j)(\mathbf{g}). \end{aligned} \quad (3.1.11)$$

Next,

$$\begin{aligned} (D_{\mathbf{f}}H_i)(\mathbf{g}) &= \frac{d}{d\epsilon} h_j(t, \mathbf{x}, f_1 + \epsilon g_1, \dots, \frac{\partial^j}{\partial t^j} \frac{\partial^{\mathbf{k}}}{\partial \mathbf{x}^{\mathbf{k}}} (f_i + \epsilon g_i), \dots) \big|_{\epsilon=0} \\ &= \frac{\partial h_j}{\partial u_1} g_1 + \dots + \frac{\partial h_j}{\partial u_i^{(j, \mathbf{k})}} \frac{\partial^j}{\partial t^j} \frac{\partial^{\mathbf{k}}}{\partial \mathbf{x}^{\mathbf{k}}} g_i + \dots \end{aligned} \quad (3.1.12)$$

The condition of infinitesimal invariance is

$$\mathbf{F}(\mathbf{f}) = 0 \Rightarrow (D_{\mathbf{f}}\mathbf{F})(\mathbf{S}(\mathbf{f})) = 0 . \quad (3.1.13)$$

The infinitesimal symmetries have the form

$$S = T \frac{\partial}{\partial t} + \sum_{i=1}^n X_i \frac{\partial}{\partial x_i} + \sum_{j=1}^m U_j \frac{\partial}{\partial u_j} \quad (3.1.14)$$

where T , X_i and U_j depend on (t, x, \mathbf{u}) . The infinitesimal action on surfaces is given by

$$\mathbf{S}(\mathbf{f}) = S_1(\mathbf{f}), \dots, S_m(\mathbf{f}) \quad (3.1.15)$$

where

$$S_i(\mathbf{f}) = -T \frac{\partial f_i}{\partial t} - \sum_{k=1}^n X_k \frac{\partial f_i}{\partial x_k} + U_i . \quad (3.1.16)$$

The way this condition is used is to replace all derivative of the form

$$\frac{\partial^{p_i+k} f_i}{dt^{p_i+k}} , \quad k \geq 0 , \quad (3.1.17)$$

that occur in $(D_{\mathbf{f}}\mathbf{F})(\mathbf{L}(\mathbf{f}))$ by the right hand side (or an appropriate derivative thereof) of the differential equation (3.1.1) . The resulting expression is called \mathbf{E} . The expression \mathbf{E} still contains solutions of the given system which need to be removed.

Now suppose that an arbitrary but finite set of values

$$u_i^{(j, \mathbf{k})} , \quad 1 \leq i \leq m , \quad 0 \leq j \leq p_i , \quad (3.1.18)$$

are given and that it is possible to find a solution \mathbf{f} of the differential equation (3.1.1) that satisfies

$$\frac{\partial^j}{\partial t^j} \frac{\partial^{\mathbf{k}}}{\partial \mathbf{x}^{\mathbf{k}}} f_i = u_i^{(j, \mathbf{k})} \big|_{t=0} . \quad (3.1.19)$$

Note that this condition is considerably weaker than requiring that (3.1.1) have a well posed initial value problem. The Cauchy-Kowalewski theorem [6] can frequently be used to show that the system of partial differential equations satisfy a considerable stronger condition. Under this condition the equation

$$\mathbf{E} = 0 \quad (3.1.20)$$

must hold with \mathbf{f} and all of its derivatives replaced by the right hand side of (3.1.1) . The resulting expression must be zero for all values of the variables

$$(t, \mathbf{x}, \mathbf{u}, \dots, u_i^{(j,\mathbf{k})}, \dots) . \quad (3.1.21)$$

Here it is important to note the coefficients of the infinitesimal symmetry do not depend on the variables $u_i^{(j,\mathbf{k})}$. Consequently the power series expansion of the expression E in the $u_i^{(j,\mathbf{k})}$ variables will not involve derivatives of the coefficients of the symmetry operator and consequently each nontrivial coefficient in the expansion will produce an equation for the coefficients of the symmetry operator. If, as is frequently the case, the expression \mathbf{E} is a polynomial in some of the variables $u_i^{(j,\mathbf{k})}$ then, the coefficients of this polynomial must be zero. When we generalize the notion of a symmetry this will no longer be true and this fact will produce one of our major difficulties.

We now turn to some examples.

3.2 The Heat Equation

The heat equation is fairly typical of partial differential equations that we would call very symmetric. Here we are discussing the heat equation in one space variable so we have two independent variables (x, t) and one dependent variable u . The heat equation is then written

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} . \quad (3.2.22)$$

It is easy to see that the heat equation has at least 5 symmetries. There are three translations, one each in x , t and u . There are two scaling symmetries, any scaling in u and scaling t with the square of a scaling in x . Because the heat equation is linear there is also an infinite symmetry group, the addition of any solution to all solutions sends the solution space into the

solution space. We will find [4] that the heat equation has two additional symmetries which are usually referred to as a hidden symmetries.

If $u = f(x, t)$ is a surface, then the operator form of the heat equation is

$$F(f) = f_t - f_{xx} \quad (3.2.23)$$

where we have used subscript notation

$$f_t = \frac{\partial f}{\partial t}, \quad f_{xx} = \frac{\partial^2 f}{\partial x^2}, \quad (3.2.24)$$

for partial derivatives. Because there is one dependent and two independent variables, the infinitesimal symmetries act on surfaces and have the form

$$S(f) = A(x, t, f)f_t + B(x, t, f)f_x + C(x, t, f) \quad (3.2.25)$$

where $A(x, t, u)$, $B(x, t, u)$ and $C(x, t, u)$ are functions that are to be determined.

Because the heat equation is linear, it is its own directional derivative. Thus

$$F(f + \epsilon g) = (f + \epsilon g)_t - (f + \epsilon g)_{xx} = f_t - f_{xx} + \epsilon(g_t - g_{xx}) \quad (3.2.26)$$

and consequently

$$(D_f F)(g) = \frac{d}{d\epsilon} F(f + \epsilon g) \big|_{\epsilon=0} = g_t - g_{xx} = F(g). \quad (3.2.27)$$

The invariance condition is then (Chapter 1 Section 6)

$$F(f) = 0 \Rightarrow (D_f F)(S(f)) = 0. \quad (3.2.28)$$

Thus, to find the conditions on A , B and C , we substitute f_{xx} for f_t in $F(S(f)) = 0$ to obtain (here A , B and C have arguments (x, t, f))

$$\begin{aligned} & A_u f_x f_{xxx} - 2A_x f_{xxx} - A_{uu} f_x^2 f_{xx} - 2B_u f_x f_{xx} \\ & - 2A_{ux} f_x f_{xx} - 2B_x f_{xx} - A_{xx} f_{xx} + A_t f_{xx} \\ & - B_{uu} f_x^3 - 2B_{ux} f_x^2 - C_{uu} f_x^2 - B_{xx} f_x \\ & + B_t f_x - 2C_{ux} f_x - C_{xx} + C_t = 0. \end{aligned} \quad (3.2.29)$$

Because the initial value problem for the heat equation is well posed, we may replace f , and the derivatives of f in the previous expression by variables, say

$$\begin{aligned} f &\rightarrow u, & f_x &\rightarrow u^1, \\ f_{xx} &\rightarrow u^{11}, & f_{xxx} &\rightarrow u^{111}. \end{aligned} \quad (3.2.30)$$

Now the expression (3.2.29) is an identity in the variables t, x, u, u^1, u^{11} and u^{111} . Because this expression is a polynomial in u^1, u^{11} and u^{111} , the coefficients of this polynomial must be zero, which yields the following set of equations:

$$\begin{aligned} A_u &= 0, \quad A_x = 0, \quad A_{uu} = 0, \quad B_{uu} = 0, \\ C_t - C_{xx} &= 0, \quad B_u + A_{ux} = 0, \quad 2B_{ux} - C_{uu} = 0, \\ 2B_x - A_{xx} + A_t &= 0, \quad 2C_{ux} - B_t + B_{xx} = 0. \end{aligned} \quad (3.2.31)$$

Note that these equations are redundant. However, there are three unknown and certainly more than three equations so they are over determined. Thus we expect the solution space to be finite dimensional and find that this is nearly true. See [56, 18] for some theorems on this point.

We now solve these equations. We note that our computer programs try to mimic, with some success, this method of solution. The first two equations in (3.2.31) give

$$A = A(t). \quad (3.2.32)$$

We should introduce a new function here, as our computer codes do, but this only makes a mess for humans to read. Plugging this back into the equations yields:

$$\begin{aligned} C_t - C_{xx} &= 0, \quad B_u = 0, \\ 2B_{ux} - C_{uu} &= 0, \quad 2B_x + A_t = 0, \\ 2C_{ux} - B_t + B_{xx} &= 0. \end{aligned} \quad (3.2.33)$$

As before, the second equation gives

$$B = B(x, t). \quad (3.2.34)$$

If the third equation is differentiated with respects to x , then

$$B_{xx} = 0. \quad (3.2.35)$$

The equation list can now be written:

$$\begin{aligned} B_{xx} &= 0, \quad C_{uu} = 0, \\ A_t + 2B_x &= 0, \quad B_t - 2C_{ux} = 0, \\ C_t - C_{xx} &= 0. \end{aligned} \tag{3.2.36}$$

Now differentiate the fourth equation in (3.2.36) twice with respects to x , then differentiate the last equation in (3.2.36) twice, once with respects to u and x and once with respects to u and t , then differentiate the fourth equation (3.2.36) with respects t and finally differentiate the third equation (3.2.36) twice with respects t twice. A little algebra the gives the following equation list:

$$\begin{aligned} A_{ttt} &= 0, \\ B_{tt} &= 0, \quad B_{xx} = 0, \\ C_{uxxx} &= 0, \quad C_{uxt} = 0, \quad C_{utt} = 0, \quad C_{uu} = 0, \\ A_t + B_x &= 0, \quad C_t - C_{ux} = 0, \quad C_t - C_{xx} = 0. \end{aligned} \tag{3.2.37}$$

Equations like the first seven equations in (3.2.37) occur frequently in symmetry calculation. This gives:

$$\begin{aligned} A &= c_1 t^2 + c_2 t + c_3, \\ B &= c_4 x t + c_5 x + c_6 t + c_7, \\ C &= R(x, t)u + h(x, t), \\ R &= c_8 t + c_9 x^2 + c_{10} x + c_{11}, \\ h_t - h_{xx} &= 0. \end{aligned} \tag{3.2.38}$$

If these results are plugged back into the equation list, then some constraints on the constants are obtained. If these constraints are solved and then the constants are relabeled, then the symmetry operator

$$\begin{aligned} S(f) &= (4k_1 t^2 + 2k_4 t + k_5) f_t \\ &\quad + (4k_1 t x + k_4 x + 2k_2 t + k_3) f_x \\ &\quad + (k_1 x^2 + k_2 x + k_1 t + k_6) f + h \end{aligned} \tag{3.2.39}$$

is obtained. Here k_1 through k_6 are arbitrary constants while h is any solution of the heat equation. The symmetry corresponding to h is obvious because

the heat equation is linear. The remaining symmetries can be translated to the vector field notation (Chapter 1 Section 4) where they form a linear space. Corresponding to each constant in (3.2.39) is a basis element. Here is a list of the basis elements.

$$\begin{aligned}
k_5 &\rightarrow \frac{\partial}{\partial t}, \text{ time translation,} \\
k_3 &\rightarrow \frac{\partial}{\partial x}, \text{ space translation,} \\
h = k_7 &\rightarrow \frac{\partial}{\partial u}, \text{ add constant to solution,} \\
k_6 &\rightarrow u \frac{\partial}{\partial u}, \text{ rescale the solution,} \\
k_4 &\rightarrow 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}, \text{ scale in time and space} \\
k_2 &\rightarrow 2t \frac{\partial}{\partial x} - xu \frac{\partial}{\partial u}, \text{ hidden symmetry} \\
k_1 &\rightarrow 4t^2 \frac{\partial}{\partial t} + tx \frac{\partial}{\partial x} - x^2 \frac{\partial}{\partial u}, \text{ hidden symmetry,}
\end{aligned} \tag{3.2.40}$$

In the file `examples.v` there are three programs, `heat_1_1`, `heat_1_2` and `heat_1_3`. The first integer in the name gives the number of spatial dimensions in the example and the second integer is a version number. The first two programs are essentially the same, `heat_1_2` prints fewer intermediate results than `heat_1_1`. The printing of the intermediate results requires a substantial amount of time because, before the results are printed they are sorted and cleaned up for the convenience of the reader. The program `heat_1_3` uses the program symmetry rather than `doitall` and consequently runs substantially faster than the other versions. However, the equation list is not completely solved so the user would normally finish this interactively. To help the user understand the interactive use of our programs we have included in `heat_1_3` the commands that we used when we solved the equations interactively. Running this program then is the same as watching the author go through an interactive session. The program `heat_1_2` uses about 23 cpu minutes on a VAX11/780 while the program `heat_1_3` uses about 11 cpu minutes. Here are listings of `heat_1_1` and `heat_1_3`.

Program heat_1_1

```
heat_1_1() := block(  
  /* The one dimensional heat equation. Use the very verbose mode so that some  
    of the inner workings of the code can be seen. This slows down the  
    program substantially. */  
  verbose : true,  
  veryverbose : true,  
  
  /* The parameter num_dif is the maximum number of terms in an equation that is  
    to be differentiated by listsolver. */  
  num_dif:4,  
  
  /* Set the dependent and independent variables. */  
  dep : [u],  
  indep : [t,x],  
  
  /* Define the differential equation. */  
  diffeqn : ['diff(u,t) - 'diff(u,x,2) = 0],  
  
  /* Now load and execute the program. */  
  load(doitall),  
  doitall(),  
  end_heat_1_1)$  
  
heat_1_3() := block(  
  /* The one dimensional heat equation. This is the same as heat_1_1 except that  
    some special tricks are used to make the code run faster. */  
  verbose : true,  
  veryverbose : false,  
  dep : [u],  
  indep : [t,x],  
  diffeqn : ['diff(u,t) - 'diff(u,x,2) = 0],  
  
  /* Not that we call symmetry and not doitall. */  
  load(symmetry),  
  symmetry(),
```



```

/* The following commands illustrate what a user might do in solving the
   determining equations for the one space dimension heat equation. */
oneterm(diff(first(eqnlist),x)),
results(),
oneterm(diff(eqnlist[2],x,2)),
results(),
eqnlist:cons(diff(last(eqnlist),u),eqnlist),
oneterm(diff(first(eqnlist),x,2)),
results(),
oneterm(diff(eqnlist[2],x)),
results(),
oneterm(diff(eqnlist[2],t)),
results(),
oneterm(diff(eqnlist[3],x,1,t,1)),
results(),
oneterm(diff(eqnlist[2],t,2)),
results(),
oneterm(diff(eqnlist[3],t)),
results(),
allnondiff(eqnlist),
results(),
end_heat_1_3)$

```

3.3 Burger's Equation

Burger's equation is a nonlinear equation similar to the heat equation,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u \frac{\partial u}{\partial x} , \quad (3.3.41)$$

where $u = f(x, t)$. There has been considerable interest in the symmetries of Burger's equation including that of the author [102]. Because (3.3.41) is nonlinear the calculation of the directional derivative of its operator form,

$$F(f) = \frac{df}{dt} - \frac{d^2 f}{dx^2} - f \frac{df}{dx} , \quad (3.3.42)$$

is interesting. Thus,

$$\begin{aligned} & \frac{d}{d\epsilon} F(f + \epsilon g) \big|_{\epsilon=0} = \\ & \frac{\partial}{\partial \epsilon} \left[\frac{\partial(f + \epsilon g)}{\partial t} - \frac{\partial^2(f + \epsilon g)}{\partial x^2} - (f + \epsilon g) \frac{\partial(f + \epsilon g)}{\partial x} \right]_{\epsilon=0} = \\ & \frac{\partial g}{\partial t} - \frac{\partial^2 g}{\partial x^2} - g \frac{\partial f}{\partial x} - f \frac{\partial g}{\partial x} . \end{aligned} \tag{3.3.43}$$

The following is a substantially edited listing of the output of a VAXIMA run that computes the symmetries of the Burger's equation.

VAXIMA Output

```
(c3) load(burgers);
Batching the file burgers.v

(c4) burgers() := block(
/* The Burger's equation. */
    verbose : true,
    veryverbose : true,

/* The parameter num_dif is the maximum number of terms in an
    equation that is to be differentiated by listsolver. */
    num_dif:7,

/* Set the dependent and independent variables. */
    dep : [u],
    indep : [t,x],

/* Define the differential equation. */
    diffeqn : ['diff(u,t) - 'diff(u,x,2) - u*'diff(u,x) = 0],

/* Now load and execute the program. */
    load(doitall),
    doitall(),
    end_burgers)$
```

Batching done.

(d5) burgers.v

(c6) burgers();

Putting the differential equations in standard form.

$$(e9) \quad \frac{du}{dt} = \frac{d^2 u}{dx^2} + u \frac{du}{dx}$$

Creating the symmetry operators.

$$l_1 = a_3 \frac{du}{dx} + a_2 \frac{du}{dt} + a_1$$

Computing the determining equations.

Eliminating the time derivatives.

Collecting the determining equations.

Calculating the coefficients of a polynomial.

Preparing the equation list for printing.

The number of equations is 9 .

$$(e12) \quad \frac{da_2}{du} = 0$$

$$(e13) \quad \frac{d^2 a_2}{du^2} = 0$$

$$(e14) \quad \frac{da_2}{dx} = 0$$

$$(e15) \quad - \frac{\frac{d^2 a2}{du^2} u}{2} - \frac{\frac{d^2 a3}{du^2}}{2} - 2 \frac{da2}{du} = 0$$

$$(e16) \quad - 2 \frac{da2}{du} u - 2 \frac{da3}{du} - 2 \frac{d^2 a2}{du dx} = 0$$

$$(e17) \quad - \frac{da1}{dx} u - \frac{d^2 a1}{dx^2} + \frac{da1}{dt} = 0$$

$$(e18) \quad - 2 \frac{d^2 a2}{du dx} u - 2 \frac{d^2 a3}{du dx} - 2 \frac{da2}{dx} - \frac{d^2 a1}{2} = 0$$

$$(e19) \quad - 3 \frac{da2}{dx} u - 2 \frac{da3}{dx} - \frac{d^2 a2}{2} + \frac{da2}{dt} = 0$$

$$(e20) \quad - \frac{da2}{dx} u^2 - \frac{da3}{dx} u - \frac{d^2 a2}{2 dx} u + \frac{da2}{dt} u - \frac{d^2 a3}{2 dx} + \frac{da3}{dt} - 2 \frac{d^2 a1}{du dx}$$

$$- a1 = 0$$

$$(e21) \quad \frac{1}{1} = a3 \frac{du}{dx} + a2 \frac{du}{dt} + a1$$

$$[a1(u, t, x), a2(u, t, x), a3(u, t, x), u(t, x)]$$

Starting the solution of the equation list.
 Collecting all one term equations.
 Length of list is 9 .
 Ordering a list of length 3 .
 Solving all one term equations.
 Fixing up a one term equation.

Attempting to solve $\frac{da2}{du} = 0$.

Fixing up a one term equation.

Attempting to solve $\frac{da4}{dx} = 0$.

The programs continues in this fashion and solves two more one term equations.
 Preparing the equation list for printing.
 The number of equations is 3 .

$$(e22) \quad \frac{da5}{dt} - 2 \frac{da6}{dx} = 0$$

$$(e23) \quad - \frac{da8}{dx} u - \frac{d^2 a8}{dx^2} u + \frac{da8}{dt} u - \frac{da7}{dx} u - \frac{d^2 a7}{dx^2} + \frac{da7}{dt} = 0$$

$$(e24) \quad - a8 u - \frac{da6}{dx} u + \frac{da5}{dt} u - 2 \frac{da8}{dx} - a7 - \frac{d^2 a6}{dx^2} + \frac{da6}{dt} = 0$$

$$(e25) \quad 1 = a6 \frac{du}{dx} + a5 \frac{du}{dt} + a8 u + a7$$

[a1(u, t, x), a2(u, t, x), a3(u, t, x), a4(x, t), a5(t),
a6(x, t), a7(x, t), a8(x, t), u(t, x)]

Differentiating the equation list for the first time.

Differentiating a list of length 3 .

Collecting all one term equations.

Length of list is 12 .

Ordering a list of length 1 .

Solving all one term equations.

Fixing up a one term equation.

$$\text{Attempting to solve } \frac{d^2 a6}{dx^2} = 0 .$$

The program now differentiates the equation list a second time and then finds enough one term equations to completely solve for all of the "a" functions. In this process constants are introduced and some of them are redundant so some of the constants are eliminated.

Collecting the non-differential equations.

Collecting coefficients

Calculating the coefficients of a polynomial.

Collecting the constants to be solved for.

Solving for the constants.

Preparing the equation list for printing.

The number of equations is 0 .

$$(e37) \quad 1 = k10 t \frac{du}{dx} x + k2 \frac{du}{dx} x + k10 x$$

$$\begin{aligned}
& + k_6 t \frac{du}{dx} + k_{11} \frac{du}{dx} + k_{10} t \frac{d^2 u}{dt} \\
& + 2 k_2 t \frac{du}{dt} + k_8 \frac{du}{dt} + k_{10} t u + k_2 u + k_6
\end{aligned}$$

(d37) end_burgers
The program has successfully found all of the symmetries of Burger's equation. Let us now use VAXIMA interactively to find a basis of the symmetry operators. d40 is the right hand side of e37.

(c43) `coeff(d40,k10);`

(d43)
$$t \frac{du}{dx} x + x + t \frac{d^2 u}{dt} + t u$$

(c44) `coeff(d40,k2);`

(d44)
$$\frac{du}{dx} x + 2 t \frac{du}{dt} + u$$

(c45) `coeff(d40,k6);`

(d45)
$$t \frac{du}{dx} + 1$$

(c46) `coeff(d40,k11);`

(d46)
$$\frac{du}{dx}$$

```
(c47) coeff(d40,k8);
```

```
(d47)          du
              --
              dt
```

This, we hope, illustrates the convenience of being able to use VAXIMA interactively to make formulas more readable.

Chapter 4

GENERALIZED SYMMETRIES

4.1 Introduction

This chapter was not completed.

In this chapter we will extend the notion of symmetry of a system of differential equations from geometric transformations to transformations that depend on the derivatives of the solution of the system. We call the most general form of these transformations, *jet* transformations. The term *jet* is borrowed from differential geometry. Many authors call this transformation Backlund or Lie-Backlund transformations. We will indicate in what sense contact transformations are a special case of jet transformations.

Because the notation becomes complicated in this theory, Section 2 begins by studying the situation in the plane. One thing to note is that there is at least an implicit choice of dependent and independent variables in this theory. Thus in three dimensions there are two distinct kinds of jet transformation, depending on whether there are two dependent and one independent or one dependent and two independent variables. To study jet transformations we must introduce infinite set of variables. In this context, it is still not completely understood how to do this in a rigorous fashion. The assumption that the various functions introduced depend on only a finite subset of the variables makes many of the objects under consideration well defined. Because this is just the assumption that makes computer programs practical, we will assume that all of the functions introduced are analytic in a finite

number of variables. In Section 3 we will study jet transformations in m dependent and n independent variables. In Section 4 we generalize the notion of symmetry.

4.2 Two Dimensional Jet Transformations

This section is devoted to motivating several definitions that are important to the theory of jet transformations. When the transformations are allowed to depend on derivatives, there are several points where confusion arises. We will clarify these points here so that we can do the general case more easily. One point is that the choice of dependent and independent variables is important. Once this choice is made, then we are always implicitly assuming the dependent variables depend on the independent variables and the derivatives under consideration are the derivatives of the dependent variables with respects to the independent variables. However, it is frequently convenient to think of the derivatives as variables in their own right. We believe that this will become apparent as we go through this section.

In the case of transformations in the plane, we assume y is dependent and x is independent, and that these variables are being transformed to ξ and η , where ξ is independent and η is dependent. In this case, the transformations that depend on one derivative have the form

$$\xi = f(x, y, \frac{dy}{dx}) , \quad \eta = g(x, y, \frac{dy}{dx}) , \quad (4.2.1)$$

where

$$f = f(x, y, v) , \quad g = g(x, y, v) , \quad (4.2.2)$$

are given functions of three variables. Here we are beginning to distinguish between the derivatives and variables that will have derivatives substituted for them. The transformation of the derivatives can be computed using the chain rule,

$$\frac{d\eta}{d\xi} = \frac{g_x dx + g_y dy + g_v dy'}{f_x dx + f_y dy + f_v dy'} = \frac{g_x + g_y y' + g_v y''}{f_x + f_y y' + f_v y''} . \quad (4.2.3)$$

If we wish to restrict ourselves to first order contact transformations, then we will require that $d\eta/d\xi$ not to depend on y'' . This can be done by requiring that

$$\frac{\partial(d\eta/d\xi)}{\partial y''} = 0 , \quad (4.2.4)$$

which means that we must have

$$g_v f_x = f_v g_x \quad \text{and} \quad g_v f_y = g_y f_v . \quad (4.2.5)$$

Now if $d\eta/d\xi$ does not depend on y'' then choosing $y'' = 0$ in (4.2.3) gives

$$\frac{d\eta}{d\xi} = \frac{g_x + g_y y'}{f_x + g_y y'} . \quad (4.2.6)$$

The definition of an n -th order contact transformation is the obvious generalization of the above ideas.

To study jet transformations we introduce the infinite set of variables x , y and \mathbf{v} where

$$\mathbf{v} = (v_1, v_2, \dots) . \quad (4.2.7)$$

As before x is independent, y is dependent and v_i is thought of as standing for the i -th derivative of y with respects to x . These variables will be transformed to ξ , η and $\boldsymbol{\nu}$ where ξ is independent, η is dependent and ν_i is thought of as standing for the i -th derivative of η with respects to ξ . Let

$$f(x, y, \mathbf{v}) \quad \text{and} \quad g(x, y, \mathbf{v}) \quad (4.2.8)$$

be functions that depend on a finite number of variables and are analytic in these variables. A *jet* transformation will have the form

$$\xi = f(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots) , \quad \eta = g(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots) . \quad (4.2.9)$$

Once the functions f and g are given, then the chain rule determines the transformation of the derivatives. The transformed derivatives will have the form

$$\frac{d^i \eta}{d\xi^i} = g_i(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots) , \quad 1 \leq i < \infty . \quad (4.2.10)$$

where, again, the g_i depend on only a finite subset of variables ξ , η and $\boldsymbol{\nu}$. If we replace the derivatives by the variables that stand for them, then the jet transformation will have the form

$$\nu_i = g_i(x, y, \mathbf{v}) , \quad 1 \leq i < \infty . \quad (4.2.11)$$

Now repeated applications of the chain rule to (4.2.9) and then replacing derivatives by the variables that stand for them gives the following form for

the g functions (here $g_0 = g$),

$$g_{i+1}(x, y, \mathbf{v}) = \frac{\frac{\partial g_i}{\partial x} + \frac{\partial g_i}{\partial y} v_1 + \sum_{k=1}^{\infty} \frac{\partial g_i}{\partial v_k} v_{k+1}}{\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} v_1 + \sum_{k=1}^{\infty} \frac{\partial f}{\partial v_k} v_{k+1}}, \quad i \geq 0. \quad (4.2.12)$$

The previous discussion was meant to motivate the following definition of a group of jet transformations. A group of jet transforms is obtained by simply letting the previous formulas depend on ϵ and then requiring the group axioms to hold. A group of jet transformations are transformations that act on the infinite set of variables

$$x, y, \mathbf{v} \quad (4.2.13)$$

and the transformations have the form

$$\begin{aligned} \xi &= f(\epsilon, x, y, \mathbf{v}) \\ \eta &= g(\epsilon, x, y, \mathbf{v}) \\ \nu_i &= g_i(\epsilon, x, y, \mathbf{v}), \quad 1 \leq i \leq \infty. \end{aligned} \quad (4.2.14)$$

The group axioms in ϵ must hold and if we set

$$f(\epsilon) = f(\epsilon, x, y, \mathbf{v}), \quad g_i(\epsilon) = g_i(\epsilon, x, y, \mathbf{v}), \quad (4.2.15)$$

then the following contact conditions must be satisfied,

$$g(\epsilon)_{i+1} = \frac{\frac{\partial g_i(\epsilon)}{\partial x} + \frac{\partial g_i(\epsilon)}{\partial y} v_1 + \sum_{k=0}^{\infty} \frac{\partial g_k(\epsilon)}{\partial v_k} v_{k+1}}{\frac{\partial f(\epsilon)}{\partial x} + \frac{\partial f(\epsilon)}{\partial y} v_1 + \sum_{k=0}^{\infty} \frac{\partial f(\epsilon)}{\partial v_k} v_{k+1}}, \quad i \geq 0. \quad (4.2.16)$$

To obtain the infinitesimal of a group of jet transformations, the previous formulas are differentiated with respects to ϵ and then ϵ is set equal to zero. Before we do the calculations we define

$$\begin{aligned} r(x, y, \mathbf{v}) &= \frac{d}{d\epsilon} f(\epsilon, x, y, \mathbf{v}) \big|_{\epsilon=0}, \\ s(x, y, \mathbf{v}) &= \frac{d}{d\epsilon} g(\epsilon, x, y, \mathbf{v}) \big|_{\epsilon=0}, \\ t_i(x, y, \mathbf{v}) &= \frac{d}{d\epsilon} g_i(\epsilon, x, y, \mathbf{v}) \big|_{\epsilon=0}. \end{aligned} \quad (4.2.17)$$

The fact that a group is the identity when $\epsilon = 0$, implies that

$$f(0, x, y, \mathbf{v}) = x, \quad g(0, x, y, \mathbf{v}) = y, \quad \nu_i(0, x, y, \mathbf{v}) = v_i. \quad (4.2.18)$$

Now differentiating (4.2.16) with respects to ϵ and setting $\epsilon = 0$ yields (here $t_0 = s$)

$$\begin{aligned} t_{i+1}(x, y, \mathbf{v}) &= \frac{\partial t_i}{\partial x} + \frac{\partial t_i}{\partial y} v_1 + \sum_{k=1}^{\infty} \frac{\partial t_k}{\partial v_k} v_{k+1} \\ &- v_{i+1} \left(\frac{\partial r}{\partial x} + \frac{\partial r}{\partial y} v_1 + \sum_{k=1}^{\infty} \frac{\partial r_k}{\partial v_k} v_{k+1} \right), \quad i \geq 0. \end{aligned} \quad (4.2.19)$$

An infinitesimal jet transformation is now a vector field

$$\mathbf{T}(x, y, \mathbf{v}) = (r(x, y, \mathbf{v}), \quad s(x, y, \mathbf{v}), \quad \mathbf{t}(x, y, \mathbf{v})) \quad (4.2.20)$$

that satisfies (4.2.19). Note that r and s determine the infinitesimal transformation and that they may be arbitrary functions of a finite number of variables. As before we can associate a first order linear partial differential operator,

$$L = r(x, y, \mathbf{v}) \frac{\partial}{\partial x} + s(x, y, \mathbf{v}) \frac{\partial}{\partial y} + \sum_{i=1}^{\infty} t_i(x, y, \mathbf{v}) \partial_i \quad (4.2.21)$$

with the vector field and that gives the infinitesimal action on functions of the variables (x, y, \mathbf{p}) .

Some of our formulas can be written more compactly if, as in [0], we introduce the operator

$$D = \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} + \sum_{k=1}^{\infty} v_{k+1} \frac{\partial}{\partial v_k}. \quad (4.2.22)$$

Then the infinitesimal contact condition (4.2.19) can be written

$$t_{i+1} = Dt_i - v_{i+1} Dr;'. \quad (4.2.23)$$

Because all of our definitions of symmetries are based on the notion of infinitesimal transformations, let us pause to make a complete definition.

Definition. Let

$$r(x, y, \mathbf{v}) \quad \text{and} \quad s(x, y, \mathbf{v}) \quad (4.2.24)$$

be analytic functions of a finite subset the countable infinity of variables

$$(x, y, \mathbf{v}) , \quad \mathbf{v} = (v_1, v_2, \dots) \quad (4.2.25)$$

and let D be defined as in (4.2.22). If we set $t_0 = s$ and then set

$$t_{i+1}(x, y, \mathbf{v}) = Dt_i(x, y, \mathbf{v}) - v_{i+1}Dr(x, y, \mathbf{v}) , \quad 0 \leq i < \infty , \quad (4.2.26)$$

then

$$\mathbf{T}(x, y, \mathbf{v}) = (r(x, y, \mathbf{v}) , \quad s(x, y, \mathbf{v}) , \quad \mathbf{t}(x, y, \mathbf{v}))' \quad (4.2.27)$$

is called a jet vector field while

$$L = r(x, y, \mathbf{v}) \frac{\partial}{\partial x} + s(x, y, \mathbf{v}) \frac{\partial}{\partial y} + \sum_{i=1}^{\infty} t_i(x, y, \mathbf{v}) \frac{\partial}{\partial v_i} \quad (4.2.28)$$

is called an infinitesimal jet operator. Both (4.2.28) and (4.2.27) are referred to as infinitesimal jet transformations.

Once we have an infinitesimal jet transformation, then we would like to find the group of jet transformations associated with the infinitesimal via the Lie series mechanism. Unfortunately, in this context, it appears that the Lie series are rarely well defined. However, this formalism is so intuitive we will continue to use it in a formal sense, that is, apply the same rules of manipulation as were valid in our previous discussions. Notice that if r and s depend on only finitely many variables, then each t_i depends on only finitely many variables. However, all of the t_i 's together depend, in general, on all of the variables. It would appear that the group action usually depends on infinitely many variables.

Let L be an infinitesimal jet transformation and then define

$$\begin{aligned} \xi(\epsilon, x, y, \mathbf{v}) &= e^{\epsilon L} x , \\ \eta(\epsilon, x, y, \mathbf{v}) &= e^{\epsilon L} y , \\ \nu_i(\epsilon, x, y, \mathbf{v}) &= e^{\epsilon L} v_i \quad 0 \leq i < \infty . \end{aligned} \quad (4.2.29)$$

Then the action of the Lie transformation on function $f(x, y, \mathbf{v})$ is given by the Composition Property,

$$e^{\epsilon L} f(x, y, \mathbf{p}) = f(\xi(x, y, \mathbf{v}) , \quad \eta(x, y, \mathbf{v}) , \quad \boldsymbol{\nu}(x, y, \mathbf{v})) .$$

The idea of action on a surface that we discussed in Chapter 1 apply to jet transformations. Thus let

$$y = g(x) \quad (4.2.30)$$

be a curve in the plane. This curve corresponds to a curve in the jet given by the infinity of equations

$$\begin{aligned} y &= g(x) , \\ v_1 &= \frac{dg}{dx}(x) , \\ v_2 &= \frac{d^2g}{dx^2} , \dots \end{aligned} \quad (4.2.31)$$

As we saw in Chapter 1, the infinitesimal action of the group on the curve was obtained by computing the action of L on the function $y - g(x)$. If we call the infinitesimal action on the curve S , then

$$S(f) = L(y - g(x)) = s(x, y, \mathbf{v}) - r(x, y, \mathbf{v}) \frac{dg}{dx}(x) . \quad (4.2.32)$$

However, the values of \mathbf{v} are given by (4.2.31), so S should, in fact, be given by

$$S(f) = -r(x, g(x), \frac{dg}{dx}(x), \dots) \frac{dg}{dx}(x) + s(x, g(x), \frac{dg}{dx}(x), \dots) . \quad (4.2.33)$$

This is the natural analog of the correspondence between infinitesimal transformations and infinitesimal actions on curves given in Chapter 1. If we write the correspondence as

$$L \rightarrow S \quad (4.2.34)$$

then it is important to notice that the correspondence is not one to one. In fact if we define a new infinitesimal by

$$\tilde{r} = 0 , \quad \tilde{s} = s - r v_1 , \quad (4.2.35)$$

then the new and old infinitesimal transformations have the same action on curves. On the other hand, if we consider the operators L such that $r \equiv 0$, then for these operators the mapping is one to one. Because we are interested in solutions of differential equations and such solutions are curves, then any two transformations that have the same action on curves will be equivalent.

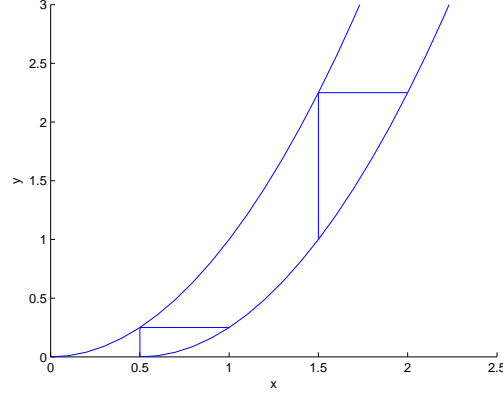


Figure 4.1: Translation Equivalence

Out of each equivalence class we prefer the transformation with $r \equiv 0$ because the formula for t_{i+1} is particularly simple.

$$t_{i+1}(x, y, \mathbf{v}) = Dt_i(x, y, \mathbf{v}) , \quad 1 \leq i < \infty . \quad (4.2.36)$$

Example. In the case of translation, it is easy to see geometrically what the above equivalence means. Figure 4.1 shows that translation of the curve $y = x^2$ by $1/2$ to the right where it becomes $y = (x - 1/2)^2$. at $x = 1/2$ horizontal and vertical lines are drawn between the first and second curves. Next these lines are moved to $x = 3/2$ so we see that the horizontal distance between the two curves has not changed, but the vertical distance is much larger. The slope of the first curve at $t = 1/2$ is $s = 1$ while at $t = 3/2$, $s = 3$. So translation to the right is equivalent to a translation downward that increases as the slope of the curve increases.

This can also be seen analytically. Let

$$L = \frac{\partial}{\partial x} , \quad (4.2.37)$$

that is,

$$r = 1 , \quad s = 0 , \quad t_i = 0 , \quad 1 \leq i < \infty . \quad (4.2.38)$$

Using the formula (2.2.36) we see that translation is equivalent to the infinitesimal vector field

$$r = 1, \quad s = -v_1, \quad t_1 = -v_2, \quad \dots, \quad (4.2.39)$$

that is, the infinitesimal operator

$$\tilde{L} = -v_1 \frac{\partial}{\partial y} - v_2 \frac{\partial}{\partial v_1} - v_3 \frac{\partial}{\partial v_2} - \dots \quad (4.2.40)$$

It is possible to compress the notation in this section slightly and this will be an advantage later. To this end let

$$v_0 = y, \quad \nu_0 = \eta, \quad g_0 = g, \quad t_0 = s. \quad (4.2.41)$$

Also, redefine the vector notation so that

$$\begin{aligned} \mathbf{v} &= (y, v_1, v_2, \dots), & \boldsymbol{\nu} &= (\eta, \nu_1, \nu_2, \dots), \\ \mathbf{g} &= (g, g_1, g_2, \dots), & \mathbf{t} &= (s, t_1, t_2, \dots). \end{aligned} \quad (4.2.42)$$

Let us redo the definition of a jet transformation using this notation.

Definition. Let

$$r(x, \mathbf{v}) \quad \text{and} \quad s(x, \mathbf{v}) \quad (4.2.43)$$

be analytic functions of a finite subset the countable infinity of variables

$$(x, \mathbf{v}), \quad \mathbf{v} = (v_0, v_1, v_2, \dots) \quad (4.2.44)$$

and let D be defined as in (4.2.22). Set

$$t_{i+1}(x, y, \mathbf{v}) = Dt_i(x, y, \mathbf{v}) - v_{i+1}Dr(x, y, \mathbf{v}), \quad 0 \leq i < \infty, \quad (4.2.45)$$

then

$$\mathbf{T}(x, y, \mathbf{v}) = (r(x, y, \mathbf{v}), \mathbf{t}(x, y, \mathbf{v})) \quad (4.2.46)$$

is called a jet vector field while

$$L = r(x, y, \mathbf{v}) \frac{\partial}{\partial x} + \sum_{i=1}^{\infty} t_i(x, y, \mathbf{v}) \frac{\partial}{\partial v_i} \quad (4.2.47)$$

is called an infinitesimal jet operator.

4.3 General Jet Transformations

A motivational discussion of jet transformations in several variables would exactly parallel the discussion in the previous section, the only difference is that there are many more indices. For this reason will not reproduce the motivation and instead we will concentrate on setting up a notation that is easily understood and helpful with our computer programs. The formulas we need are “obvious” generalizations of the formulas in the previous section.

We will consider the situation in which there are n independent variables and m dependent variables,

$$\mathbf{x} = (x_1, x_2, \dots, x_n), \quad \mathbf{u} = (u_1, u_3, \dots, u_m).$$

A notation will be needed for derivatives of all orders,

$$\begin{aligned} u_i^{\{0\}} &\Longleftrightarrow u_i, \\ u_i^{\{j\}} &\Longleftrightarrow \frac{\partial u_i}{\partial x_j}, \\ u_i^{\{j,k\}} &\Longleftrightarrow \frac{\partial^2 u_i}{\partial x_j \partial x_k}, \\ u_i^{\{j,k,\ell\}} &\Longleftrightarrow \frac{\partial^3 u_i}{\partial x_j \partial x_k \partial x_\ell}, \\ &\dots \end{aligned} \tag{4.3.48}$$

The superscripts will soon become outrageous so we introduce a notation for them,

$$\boldsymbol{\sigma} = \{\sigma_1, \sigma_2, \dots, \sigma_k\}, \tag{4.3.49}$$

where σ_i is an integer, $1 \leq \sigma_i \leq n$ and the sequence $\boldsymbol{\sigma}$ may be of arbitrary length. Then

$$u_i^{\boldsymbol{\sigma}} \Longleftrightarrow \frac{\partial^k u_i}{\partial x_{\sigma_1} \dots \partial x_{\sigma_k}}. \tag{4.3.50}$$

Because it is possible to interchange the order of partial differentiation, the sequences will be required to be nondecreasing,

$$0 < \sigma_1 \leq \sigma_2 \leq \dots \leq \sigma_k. \tag{4.3.51}$$

The jet space will consist of the infinite set of variables

$$\mathbf{v} = (\mathbf{x}, \mathbf{u}, \dots, u_i^{\boldsymbol{\sigma}}, \dots). \tag{4.3.52}$$

It will also be helpful to introduce an operation on the sequences, which is written as $\{\boldsymbol{\sigma}, i\}$ where $1 \leq i \leq n$. The value of $\{\boldsymbol{\sigma}, i\}$ is a sequence with one more term than $\boldsymbol{\sigma}$ obtained by placing i in $\boldsymbol{\sigma}$ in such a way that the result, $\{\boldsymbol{\sigma}, i\}$, remains nondecreasing.

One of the things that we use to help clarify dependency problems are the *chain rule operators*,

$$D_i = \frac{\partial}{\partial x_i} + \sum_{k=1}^m u_k^{\{i\}} \frac{\partial}{\partial u_k} + \sum_{k=1, s \neq 0}^m u_k^{\{s, i\}} \frac{\partial}{\partial u_k^s}, \quad 1 \leq i \leq n. \quad (4.3.53)$$

In certain formulas, it is not clear whether differentiation is being performed before or after substitution. The chain rule operators will clarify this situation. They will also be useful in the computer programs. Let $F(\boldsymbol{v})$ be a function on the jet and $\boldsymbol{u} = \boldsymbol{f}(\boldsymbol{x})$ be a hyper surface, that is, a mapping of the independent to the dependent variables. We will use the same notation as before for the derivatives of \boldsymbol{f} ;

$$f_i^\sigma, \Longleftrightarrow \frac{\partial^k f_i}{\partial x_{\sigma_1} \dots \partial x_{\sigma_k}}. \quad (4.3.54)$$

Now it is possible to replace the jet variables with the derivatives for which they stand;

$$F(\boldsymbol{f}(\boldsymbol{x})) \equiv F(\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x}), \dots, f_i^\sigma(\boldsymbol{x}), \dots). \quad (4.3.55)$$

With all of the notation defined, the chain rule can be expressed succinctly as follows.

Proposition. Let $F(\boldsymbol{v})$ and $\boldsymbol{f}(\boldsymbol{x})$ be given. Then

$$\frac{\partial}{\partial x_i}(F(\boldsymbol{f}(\boldsymbol{x}))) = D_i F(\boldsymbol{v})|_{\boldsymbol{v}=\boldsymbol{f}}. \quad (4.3.56)$$

An infinitesimal jet vector field is given by

$$\boldsymbol{T} = (\boldsymbol{r}(\boldsymbol{v}), \boldsymbol{s}(\boldsymbol{v}), \dots, t_i^\sigma(\boldsymbol{v}), \dots) \quad (4.3.57)$$

where

$$\boldsymbol{r}(\boldsymbol{v}) = (r_1(\boldsymbol{v}), \dots, r_n(\boldsymbol{v})), \quad \boldsymbol{s}(\boldsymbol{v}) = (s_1(\boldsymbol{v}), \dots, s_m(\boldsymbol{v})), \quad (4.3.58)$$

and the conditions on \mathbf{r} , $\boldsymbol{\sigma}$ and $t_i^\sigma(\mathbf{v})$ are given below. An infinitesimal jet operator is given by

$$L = \sum_{i=1}^n r_i(\mathbf{v}) \frac{\partial}{\partial x_i} + \sum_{i=1}^m s_i(\mathbf{v}) \frac{\partial}{\partial u_i} + \sum_{i=1, \sigma \neq 0}^m t_i^\sigma(\mathbf{v}) \frac{\partial}{\partial u_i^\sigma} \quad (4.3.59)$$

where

$$\begin{aligned} r_i(\mathbf{v}) , \quad 1 \leq i \leq n , \\ s_i(\mathbf{v}) , \quad 1 \leq i \leq m , \\ t_i^\sigma(\mathbf{v}) , \quad 1 \leq i \leq m , \quad \boldsymbol{\sigma} \neq 0 , \end{aligned} \quad (4.3.60)$$

are real analytic functions of a finite subset of the variables \mathbf{v} . We introduce the special superscript $\{0\}$ and define $\{\{0\}, i\} = \{i\}$ and then if we set

$$t_i^{\{0\}} = u_i , \quad 1 \leq i \leq m . \quad (4.3.61)$$

Now the t_i^σ are defined recursively by

$$t_i^{\{\sigma, k\}} = D_k(t_i^\sigma) - \sum_{j=1, \sigma \neq 0}^n u_j^{\{\sigma, \ell\}} D_k(s_\ell) , \quad 1 \leq i \leq n . \quad (4.3.62)$$

We can write the group action associated with an infinitesimal jet operator as a formal Lie series:

$$\boldsymbol{\xi}(\epsilon) = e^{\epsilon L} \mathbf{x} , \quad (4.3.63)$$

$$\boldsymbol{\eta}(\epsilon) = e^{\epsilon L} \mathbf{u} , \quad (4.3.64)$$

$$\nu_i^\sigma(\epsilon) = e^{\epsilon L} = u_i^\sigma .$$

The Composition Property of Lie series tells us that

$$e^{\epsilon L} F(\mathbf{v}) = g(e^{\epsilon L} \mathbf{x} , e^{\epsilon L} \mathbf{u}, \dots, e^{\epsilon L} u_i^\sigma, \dots) . \quad (4.3.65)$$

This identity will be useful when we study the invariance of differential equations under jet transformations.

Before we can formulate the invariance of differential equations we need to know how infinitesimal jet transformations operate on surfaces (hyper

surfaces). If $\mathbf{u} = \mathbf{f}(\mathbf{x})$ is a surface then this surface corresponds to a surface in the jet variable \mathbf{v} given by

$$u_i^\sigma = \frac{\partial^k f_i}{\partial x_{\sigma_1} \dots \partial x_{\sigma_k}}(\mathbf{x}) . \quad (4.3.66)$$

It was seen Chapter 1 that if L is an infinitesimal jet operator, then the action on surfaces is given by

$$\begin{aligned} \mathbf{S}(\mathbf{f}) &= L(\mathbf{u} - \mathbf{f}(\mathbf{x})) \\ &= \mathbf{s}(\mathbf{v}) - \sum_{i=1}^n r_i(\mathbf{v}) \frac{\partial \mathbf{f}}{\partial x_i}(\mathbf{x}) = -(\mathbf{r}(\mathbf{v}) \cdot \nabla_x) \mathbf{f}(\mathbf{x}) + \mathbf{s}(\mathbf{v}) . \end{aligned} \quad (4.3.67)$$

The \mathbf{v} arguments are now constrained by (4.3.67), so the final form for the action on surfaces is

$$\mathbf{S}(\mathbf{f}) = -\mathbf{r}(\mathbf{f}(\mathbf{x})) \cdot \nabla_x \mathbf{f}(\mathbf{x}) + \mathbf{s}(\mathbf{f}(\mathbf{x})) . \quad (4.3.68)$$

This agrees , of course, with the formulas derived in Chapter 1.

It is not possible to give an explicit formula of the exponential

$$e^{\epsilon \mathbf{S}} \mathbf{g}(\mathbf{x}) . \quad (4.3.69)$$

However, this action can be described implicitly. One way of doing this is to write

$$\mathbf{g}(\epsilon, \mathbf{x}, \mathbf{u}) = e^{\epsilon L}(\mathbf{u} - \mathbf{f}(\mathbf{x})) \quad (4.3.70)$$

and the solve

$$\mathbf{g}(\epsilon, \mathbf{x}, \mathbf{u}) = 0 \quad (4.3.71)$$

for the transformed surface,

$$\mathbf{u} = \mathbf{f}(\epsilon, \mathbf{x}) . \quad (4.3.72)$$

It is also possible to specify the evolution of the surface by an initial value problem for an infinite order partial differential equation,

$$\frac{\partial \mathbf{f}(\epsilon \mathbf{x})}{\partial \epsilon} = \mathbf{S}(\mathbf{x}, \mathbf{f}(\epsilon, \mathbf{x})) , \mathbf{f}(0, \mathbf{x}) = \mathbf{f}(\mathbf{x}) . \quad (4.3.73)$$

Remark. It will be an unusual situation when the initial value problem for

$f(\epsilon, x)$ given in the previous proposition will be well posed, so Lie Backlund transformation will not generate a well defined group motion on surfaces.

As before, there is a notion of equivalence of infinitesimal jet transformations.

Definition. Two infinitesimal jet transformations are said to be equivalent, $L \simeq \tilde{L}$, when the operators L and \tilde{L} produce the same action on surfaces.

It is also important to know when jet transformations are, in fact, equivalent to a simpler transformation.

Proposition. A jet transformation L of the form (4.3.59) is equivalent to a point transformation if and only if (4.3.58)

$$\mathbf{r}(\mathbf{v}) - \mathbf{s} \quad (4.3.74)$$

is linear in $v_i^{\{k\}}$.

Proof. An infinitesimal point transformation is equivalent to

$$L = \sum_{i=1}^n r_i(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x_i} + \sum_{i=1}^m s_i(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u_i} + \sum_{i=1}^m t_i^\sigma(\mathbf{v}) \frac{\partial}{\partial u_i^\sigma} \quad (4.3.75)$$

where t_i^σ are defined in the usual way. If the conditions of the proposition hold then the infinitesimal jet symmetry is equivalent to an infinitesimal of the previous form.

Proposition. A jet transformation L is equivalent to a first order contact transformation if and only if (4.3.58) $\mathbf{r}(\mathbf{v}) - \mathbf{s}$ depends only on \mathbf{x} , \mathbf{u} and $u_i^{\{k\}}$.

Proof. For such a P to generate a contact transformation we must have

$$P = \eta(x, u, u_1, \dots, u_n) - \sum u_i \xi_i(x, u, u_1, \dots, u_n) .$$

4.4 Invariance of Differential Equations

The derivation of the conditions that describe the invariance of a system of differential equation under a group of jet transformations is easy now that we

have done the ground work in the previous section. Also the introduction of the jet variables makes the description of a system of partial differential equations easy. Thus, \mathbf{F} is a system of ℓ partial differential equations provided that

$$\mathbf{F}(\mathbf{v}) = (F_1(\mathbf{v}), \dots, F_\ell(\mathbf{v})) \quad (4.4.76)$$

where F_i , $1 \leq i \leq \ell$ are analytic functions of a finite number of the jet variables \mathbf{v} . A surface (function) $\mathbf{u} = \mathbf{f}(\mathbf{x})$ is a solution of the system of partial differential equations provided that

$$\mathbf{F}(\mathbf{f}(\mathbf{x})) = 0 . \quad (4.4.77)$$

Theorem. The solution space of the differential equation

$$\mathbf{F}(\mathbf{g}(\mathbf{x})) = 0$$

is invariant under the group of jet transformation $\exp(\epsilon L)$ if and only if

$$L(\mathbf{F}(\mathbf{v})) = 0$$

when the infinite set of conditions written below hold:

$$\mathbf{F}(\mathbf{v}) = 0 , \ D_i \mathbf{F}(\mathbf{v}) = 0 , \ D_i D_j (\mathbf{F}(\mathbf{v})) = 0 , \ \dots$$

REFERENCES

The references are divided into three groups: the first group consists of books, monographs, reviews and papers that should be of general interest; the second group contains references to the computer symbol manipulation literature while the the third group contains references to the research literature. We have also included some items of historical interest in the first group. The references to the research literature are by no means complete. We have attempted to provide a large sample of the recent research literature. No coverage has been give to many related topics including similarity methods in engineering, Lie group and symmetry methods in physics and the currently active mathematical area of differential equations on Lie groups. We have tried to emphasize works that consider the differential equations to be the important given object and then proceed to study the related symmetries, groups and algebras.

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Appendix A

A.1 One Term PDE's

We present a method for solving equations of the form

$$\frac{\partial^{p_1}}{\partial x_1^{p_1}} \frac{\partial^{p_2}}{\partial x_2^{p_2}} \cdots \frac{\partial^{p_n}}{\partial x_n^{p_n}} f(x_1, \dots, x_n) = 0 \quad (\text{A.1.1})$$

will be described. Here $p_i \geq 0$. Recall that repeated integration gives the general solution of the one variable equation

$$\frac{d^j}{dx^j} f(x) = 0 \quad (\text{A.1.2})$$

as

$$f(x) = \sum_{i=0}^{j-1} a_i x^i, \quad (\text{A.1.3})$$

where $a_i, 0 \leq i \leq j-1$ are constants. The multivariable case is a bit more complicated.

Proposition. The general solution of the partial differential equation (A.1.1) is

$$f(x_1, \dots, x_n) = \sum_{k=1}^n P_k, \quad (\text{A.1.4})$$

where

$$P_k = \sum_{i=0}^{k-1} a_i^k x_k^i \quad (\text{A.1.5})$$

and a_i^k is an arbitrary function of all of the variables (x_1, \dots, x_n) except x_k .

Proof. If all of the $p_i = 0$ then the differential equation is trivial and the formula (A.1.5) gives the correct solution, $f \equiv 0$. Now proceed by induction. Assume that (A.1.5) is true and then try to solve

$$\frac{\partial^{p_1+1}}{\partial x_1^{p_1+1}} \frac{\partial^{p_2}}{\partial x_2^{p_2}} \cdots \frac{\partial^{p_n}}{\partial x_n^{p_n}} \tilde{f} = 0. \quad (\text{A.1.6})$$

If we set $f = \partial \tilde{f} / \partial x_1$, then f satisfies (A.1.1) and then the induction hypothesis says that f is given by (A.1.4).

If P_k is one of the terms in (A.1.5) and P_k is antidifferentiated with respects to x_1 , then two possible things happen. If $k \neq 1$, then the antiderivative of P_k has exactly the same form as P_k with new coefficients \tilde{a}_i^k . If $k = 1$ then

$$\int P_k dx_1 = \sum_{i=0}^{p_k-1} a_i^k \frac{x_1^{i+1}}{i+1} + a \quad (\text{A.1.7})$$

where a depends on all of the variables except x_1 . A change of summation index gives, as was desired,

$$\int P_k dx_1 = \sum_{i=0}^{p_k} \tilde{a}_i^k x_1^i, \quad (\text{A.1.8})$$

where $\tilde{a}_i^0 = a$ and $\tilde{a}_i^{k+1} = a_i^k$, or $1 \leq i \leq p_k$. Because it is possible to interchange the order of partial differentiation, the argument for x_1 works for any x_j and consequently the induction is finished.